Quantum Chromodynamics

Lecture 3: The strong coupling and pdfs

Hadron Collider Physics Summer School 2010

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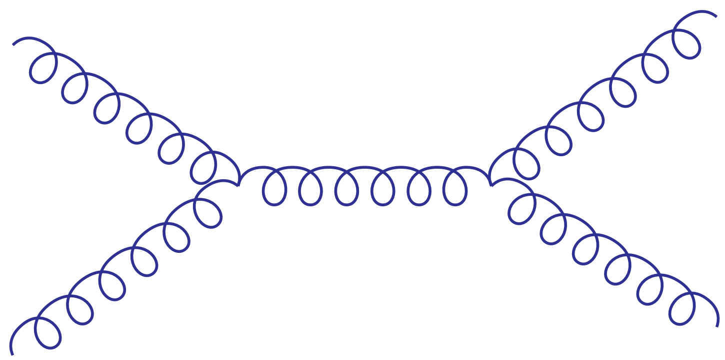
Tasks for today

- Understand the need for renormalization.
  - ultraviolet singularities and the running coupling.
- Understand the importance of factorization.
  - overview of parton distribution functions.
- Investigate some phenomenological consequences of the renormalization and factorization procedures.
  - motivation for higher orders in perturbation theory.

A simple loop integral

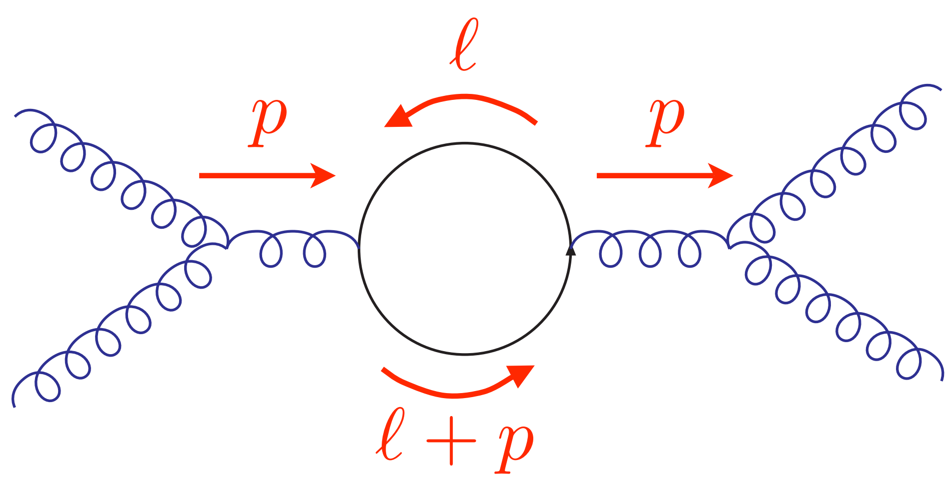
- Take a very simple process at hadron colliders - inclusive jet production.

example diagram for gg→gg



$$\text{amplitude} \sim g_s^2 \sim \alpha_s$$

- Now consider higher order perturbative corrections to this process.
  - if we don't want to change final state all we can do is add internal loops, e.g.



$$\text{amplitude} \sim (g_s^2)^2 \sim \alpha_s^2$$

Feynman rules: integrate over unconstrained loop momentum:

$$\int \frac{d^4\ell}{(2\pi)^4} \frac{1}{\ell^2(\ell+p)^2}$$

Regularization

- For large loop momenta we have a problem:

$$\int \frac{d^4\ell}{(2\pi)^4} \frac{1}{\ell^2(\ell+p)^2} \sim \frac{1}{(2\pi)^4} \int \frac{|\ell|^3 d|\ell|}{(\ell^2)^2} \sim \log(|\ell|)$$

- This is called an ultraviolet singularity.
- Regularization is the procedure with which we handle this singularity.
- Obvious solution: cut off all loop integrals at some scale Λ with the singularities all now manifest as terms proportional to log(Λ) .
  - main problem: not gauge invariant.
- The usual solution nowadays is to use dimensional regularization: change from the normal 4 to d=4-2ε dimensions.

$$\int \frac{d^{4-2\epsilon}\ell}{(2\pi)^{4-2\epsilon}} \frac{1}{\ell^2(\ell+p)^2} \sim \frac{1}{(2\pi)^{4-2\epsilon}} (p^2)^{-\epsilon} \int \frac{d|\ell|}{|\ell|^{1+2\epsilon}} \sim \frac{(p^2)^{-\epsilon}}{\epsilon}$$

this must be the factor, by dimension counting

for $\epsilon>0$, i.e. less than 4 dim.

Renormalization

- QCD is a renormalizable theory, which means that these singularities can be absorbed into a small number of (infinite) bare quantities.
  - any physical observable, computed using the renormalized quantities, is then finite.
- In dimensional regularization, we changed the dimensionality of our integral in order to render it finite. In order to keep physical observables in four dimensions we must introduce a quantity to absorb the extra dimensions, i.e.

$$\frac{(p^2)^{-\epsilon}}{\epsilon} \longrightarrow \frac{(p^2/\mu^2)^{-\epsilon}}{\epsilon} = \frac{1}{\epsilon} - \log(p^2/\mu^2)$$

- The new quantity μ is the renormalization scale. Renormalized quantities depend on μ.
- The singularity is now easily removed by subtraction, but there is ambiguity in whether any constant (if any) also goes.

just the pole	minimal subtraction (MS)
pole + specific constant	$\overline{\text{MS}}$ ("MS-bar")

Renormalization scale independence

- For a meaningful theory, it must be that any physical observable *R* is independent of the (arbitrary) choice of μ.
- Choose particular observable that depends on a single hard energy scale, *Q*, (e.g. inclusive W production at the LHC: $Q = M_W$).
- This observable can only depend upon the ratio of the dimensionful scales, *Q/μ*, and on the renormalized coupling, $\alpha_s \equiv \alpha_s(\mu)$.

$$\frac{dR}{d\mu} = \left(\frac{\partial}{\partial\mu} + \frac{\partial\alpha_s}{\partial\mu} \frac{\partial}{\partial\alpha_s} \right) R = 0$$

$$\Longrightarrow \left(-(Q/\mu^2) \frac{\partial}{\partial(Q/\mu)} + \frac{\partial\alpha_s}{\partial\mu} \frac{\partial}{\partial\alpha_s} \right) R = 0$$

- Two definitions help to simplify this:

$$t = \log(Q/\mu)$$

(recognizes logarithmic derivative in first term)

$$\beta(\alpha_s) = \mu \frac{\partial\alpha_s}{\partial\mu}$$ beta function

(parametrizes unknown in second term)

The running coupling

$$\left(-\frac{\partial}{\partial t} + \beta(\alpha_s)\frac{\partial}{\partial \alpha_s}\right) R(e^t, \alpha_s) = 0$$

- This has a simple solution if we allow a running coupling, $\alpha_s(Q)$.
- In that case, we can balance the partial derivatives by requiring,

$$\beta(\alpha_s) = \frac{\partial \alpha_s}{\partial t} \qquad \Longrightarrow \qquad t = \int_{\alpha_s}^{\alpha_s(Q)} \frac{dx}{\beta(x)}$$

- Differentiating this form of the solution then gives the further relation:

$$\beta(\alpha_s(Q)) = \frac{\partial \alpha_s(Q)}{\partial t} \qquad \Longrightarrow \qquad \frac{\partial \alpha_s(Q)}{\partial \alpha_s} = \frac{\beta(\alpha_s(Q))}{\beta(\alpha_s)}$$

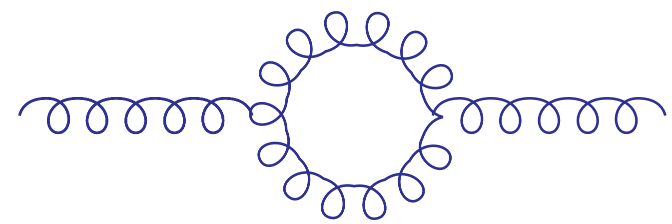
- These two identities ensure that the function $R(1, \alpha_s(Q))$ is also a solution.

$R(Q/\mu, \alpha_s)$ dependence on ren. scale and bare coupling	$\Longrightarrow$	$R(1, \alpha_s(Q))$ renormalized coupling, scale dependence in α_s

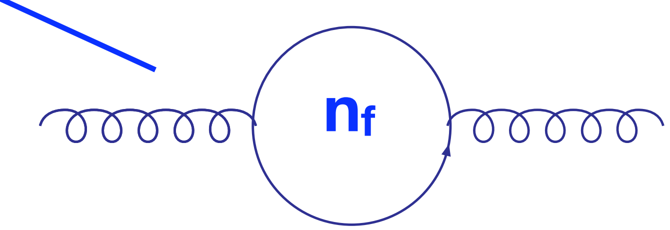
The beta function

$$\beta(\alpha_s) = \mu\,\frac{\partial \alpha_s}{\partial \mu} = \frac{\partial \alpha_s}{\partial(\log \mu)}$$

- The beta function must be extracted from higher order loop calculations, i.e. in a perturbative fashion.
- At one-loop we find:



$$\beta(\alpha_s) = -b_0\,\alpha_s^2 + \dots\,, \qquad b_0 = \frac{11N_c - 2n_f}{6\pi}$$



- In QCD, the beta-function is negative.
  - this is in contrast to QED, where there is no color term, so positive.
- The beta function of QCD has now been computed up to 4 loops
  - further perturbative corrections do not change the essential features of this picture.

Explicit running

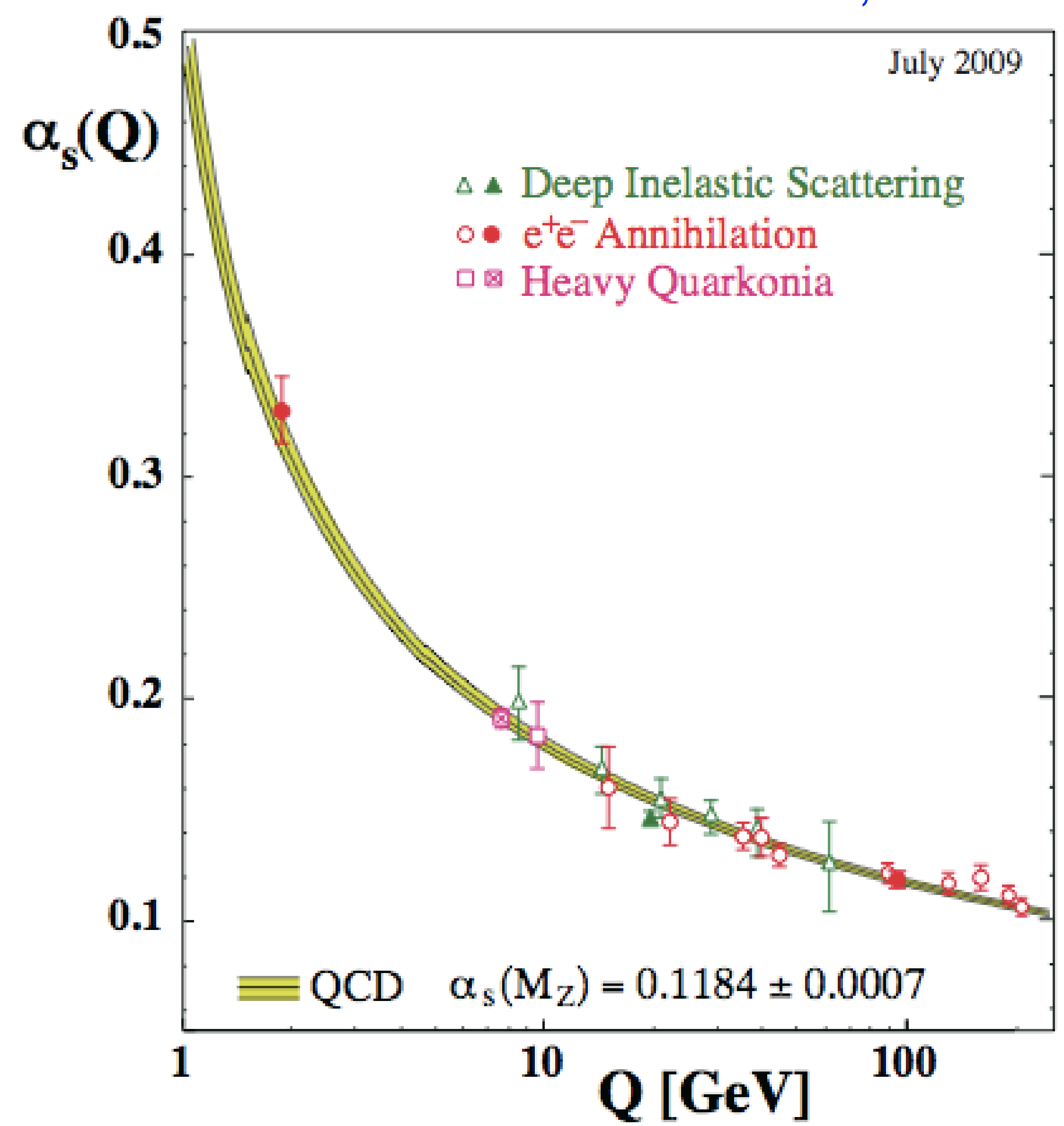
- With this perturbative form, can now solve for the form of the running coupling.

$$\frac{\partial \alpha_s}{\partial(\log\mu)} = -b_0 \alpha_s^2 \implies \left[\frac{1}{\alpha_s}\right]^{\mu=Q} = b_0 \left[\log\mu\right]^{\mu=Q}$$

$$\implies \alpha_s(Q) = \frac{\alpha_s(\mu)}{1+\alpha_s(\mu) b_0 \log(Q/\mu)}$$

- As Q increases, the denominator wins and the coupling goes to zero.
  - this is asymptotic freedom.
- In the opposite limit the coupling becomes large.
  - our perturbation theory is no longer reliable.
  - suggests onset of confinement (not yet demonstrated in QCD).

S. Bethke, 2009



Conventions

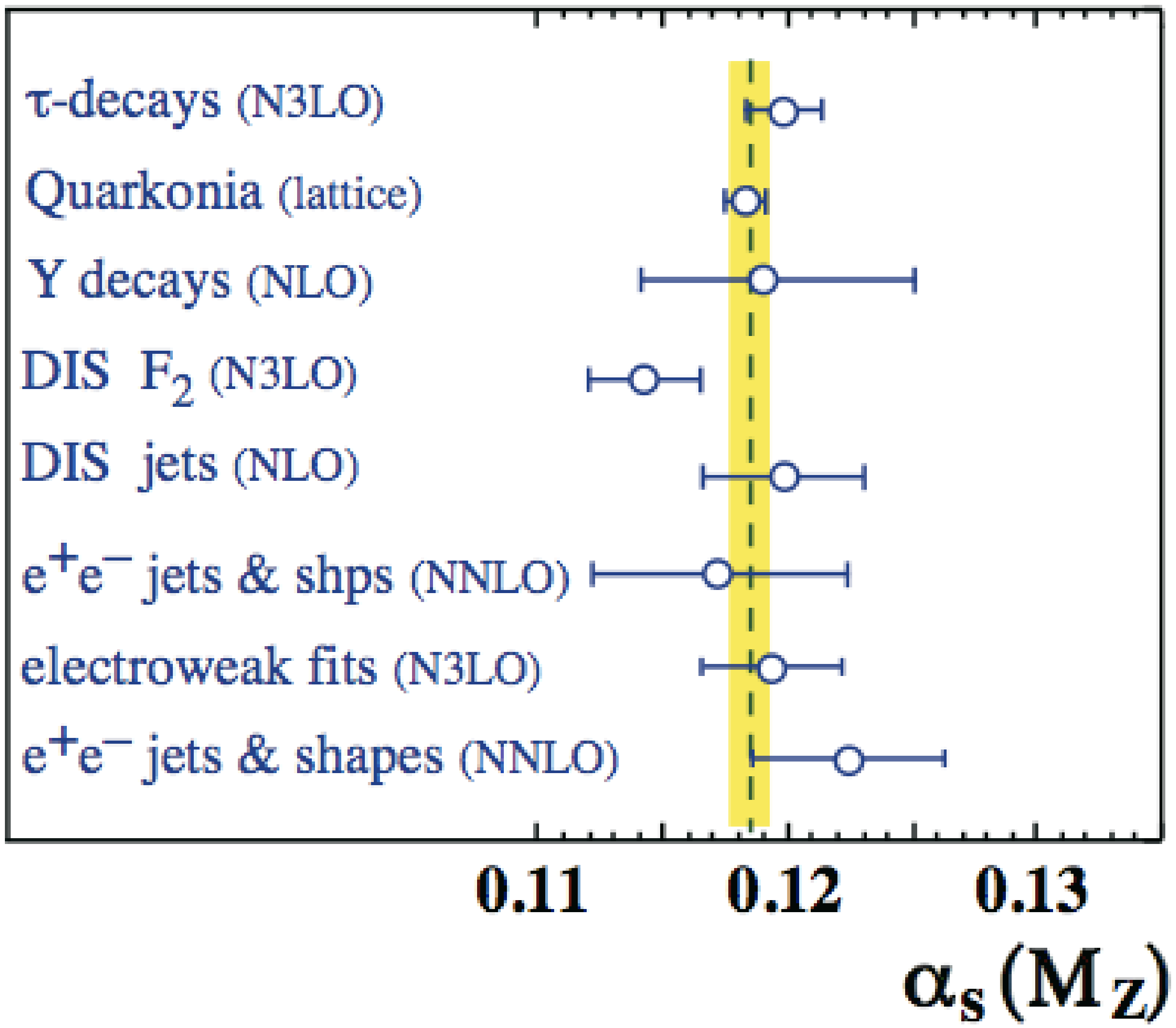
- It used to be common to write equations for the running coupling in terms of a parameter Λ_{QCD} - roughly, the scale at which the coupling becomes large. At one loop:

$$\alpha_s(Q) = \frac{\alpha_s(\mu)}{1 + \alpha_s(\mu) b_0 \log(Q/\mu)} \longrightarrow \frac{1}{b_0 \log(Q/\Lambda_{QCD})}$$

$$\Longrightarrow \Lambda_{QCD}^{(1-\text{loop})} = Q \exp\left[-1/(b_0 \alpha_s(Q))\right]$$

- Measurements of the strong coupling suggest Λ_{QCD} in the range 200-300 MeV.
- Unfortunately the definition of Λ_{QCD} must change when working at higher orders and when including different numbers of light flavors → confusion!
- A better - and now widespread - convention is to refer to the strong coupling at a reference scale, usually M_Z.
  - far away from quark thresholds, well into the perturbative region
  - convenient for the many measurements taken on the *Z* pole at LEP.

Determinations of $\alpha_s(M_Z)$

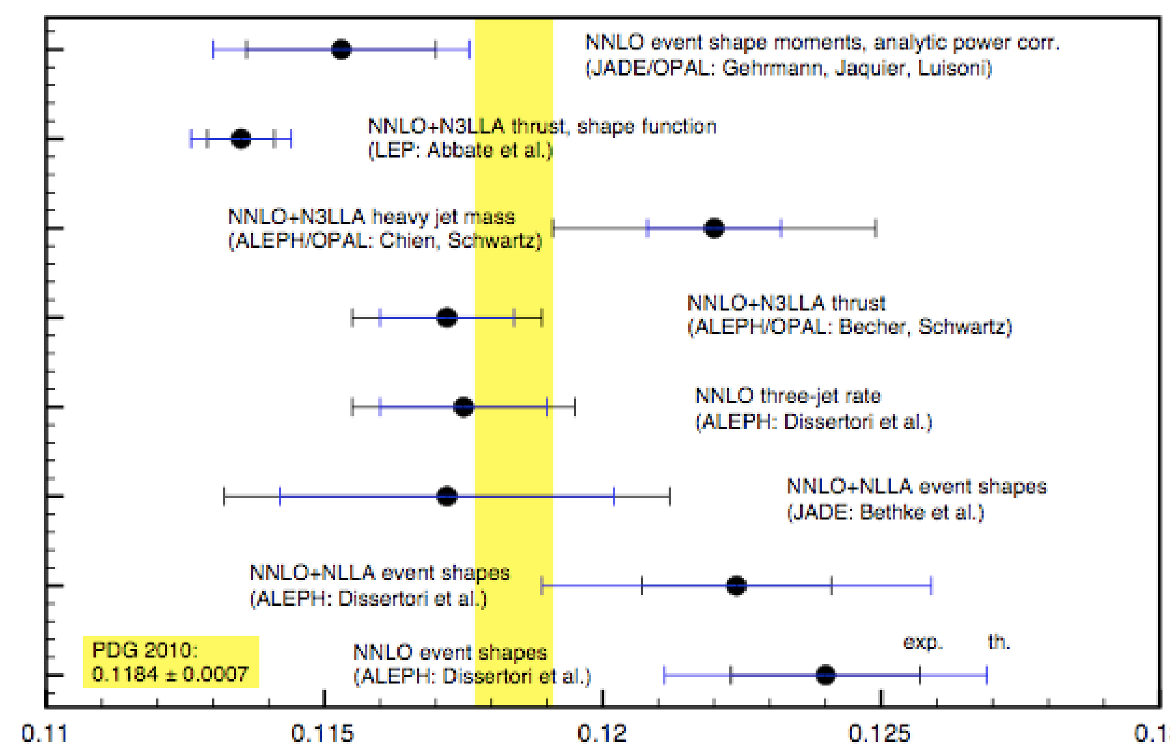


S. Bethke, 2009

- Broad agreement between different extractions
  - many different experiments with (mostly) unrelated sources of error.

$$\alpha_s(M_Z) = 0.1184 \pm 0.0007$$

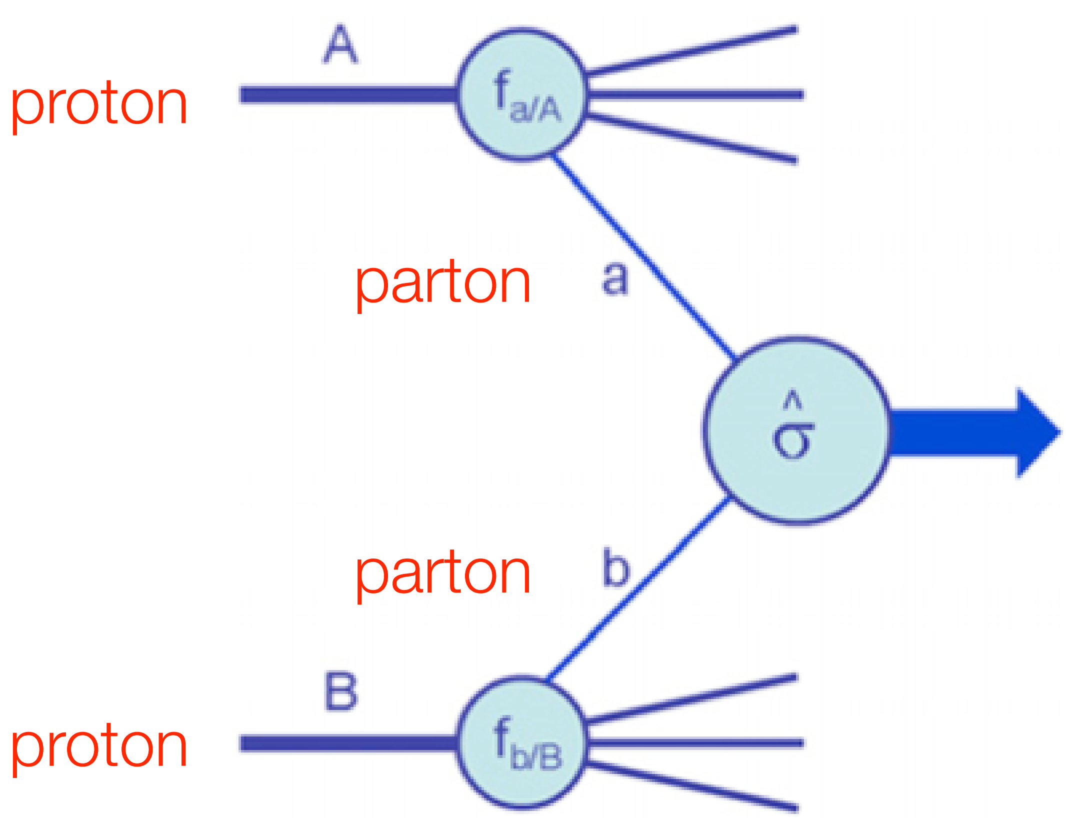
- Some signs that very recent determinations from event shapes at LEP may be consistently smaller than low-energy extractions.



Partons and protons

- An important consideration, that we have not yet discussed, is that we are in the era of hadron colliders.
- We have already seen that the QCD Lagrangian tells us how to describe QCD in terms of partons, but struggles with hadrons.

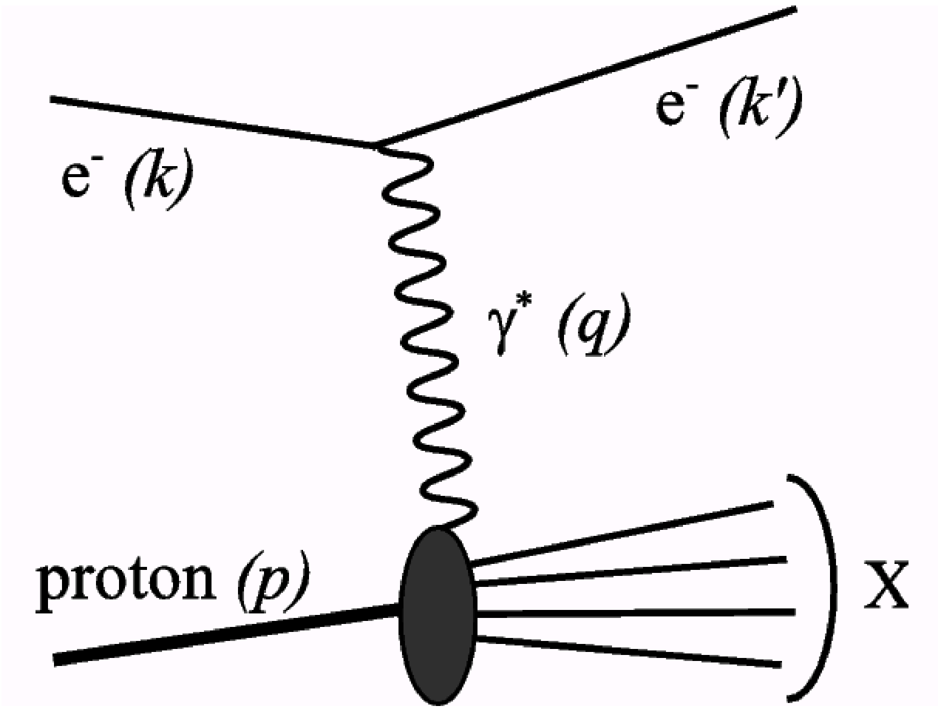
$$\sigma_{AB} = \int dx_a dx_b \, f_{a/A}(x_a) \, f_{b/B}(x_b) \, \hat{\sigma}_{ab \to X}$$



- A "simple" formalism can be introduced to help.
- It describes the cross section in terms of a factorization:
  - soft physics describing the probability of finding, within a proton, a parton with a given momentum fraction of proton.
  - a subsequent hard scattering between partons (well-described in QCD pert. th.)

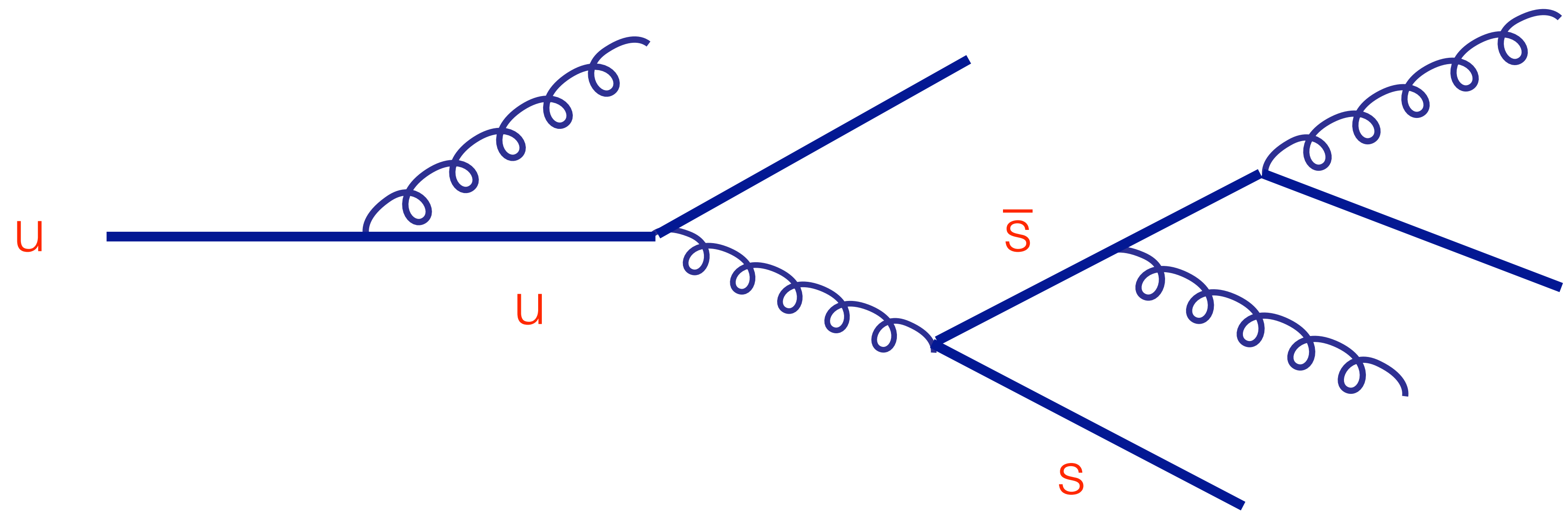
Parton distribution functions

- The "probability" functions are parton distribution functions (pdfs): $f_{a/A}(x_a)$.
  - in this simple picture, they are functions of momentum fraction, $x_a = E_a/E_P$.
- Since they cannot be computed from first principles, they must be extracted from experimental data.
- Deep inelastic scattering in ep collisions (HERA) is an ideal environment in which to do this.
  - pdf (QCD) enters only in part of the initial state;
  - the rest is QED - well-known.
- Valence quark distributions are the obvious ones. For a proton, u and d.
- Sea quarks are the rest, which one can think of as being produced from gluon splitting inside the proton.



QCD-improved parton model

- How likely are we to find such a sea quark, with a given momentum fraction?



- The answer of course depends upon how many such branchings have occurred within the proton before the hard scattering takes place.
- If this looks familiar, it is - the picture is very much the same as for parton showers in the final state.
- The formalism leads to a picture in which the pdfs must also be functions of the scale at which they are probed: $f(x) \to f(x, Q^2)$, together with a DGLAP equation as before:

$$Q^2 \frac{\partial f(x,Q^2)}{\partial Q^2} = \int_0^1 dz \left(\frac{\alpha_s}{2\pi}\right) P_{ab}(z) \left(\frac{1}{z} f(x/z, Q^2) - f(x, Q^2)\right)$$

DGLAP revisited

- In this context, the equation is more usefully written in the form:

$$\frac{\partial f(x,Q^2)}{\partial \log Q^2} = \int_0^1 \frac{dz}{z} \left(\frac{\alpha_s}{2\pi}\right) [P_{ab}(z)]_+ \, f(x/z, Q^2)$$

where the new "plus prescription" is defined by:

$$\int_0^1 dz\, g(z)_+ f(z) = \int_0^1 dz\, g(z) \left[f(z) - f(1)\right]$$

- We see that, written in this way, it is clear there can be no singularity as $z \to 1$.

$$\begin{aligned}
\int_0^1 [P_{qq}(z)]_+ \, f(z) &= C_F \int_0^1 dz \left(\frac{1+z^2}{1-z}\right) [f(z) - f(1)] \\
&= C_F \int_0^1 dz \left\{ \left(\frac{1+z^2}{1-z}\right) f(z) - \left[\frac{2}{1-z} - (1+z)\right] f(1) \right\} \\
&= C_F \int_0^1 \frac{dz}{1-z} \left\{(1+z^2) f(z) - 2f(1)\right\} + \frac{3}{2} C_F f(1) \\
&= C_F \int_0^1 dz \left\{ \frac{1+z^2}{(1-z)_+} + \frac{3}{2}\delta(1-z) \right\} f(z)
\end{aligned}$$

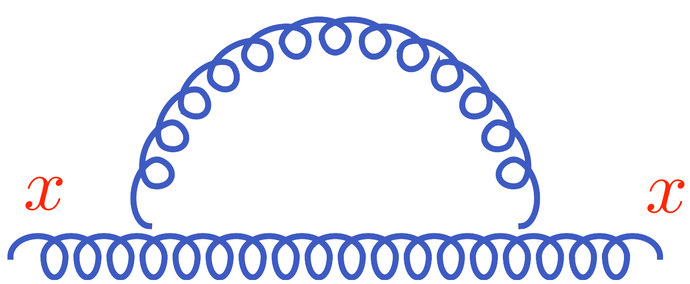
Regularized splitting functions

- We have found:

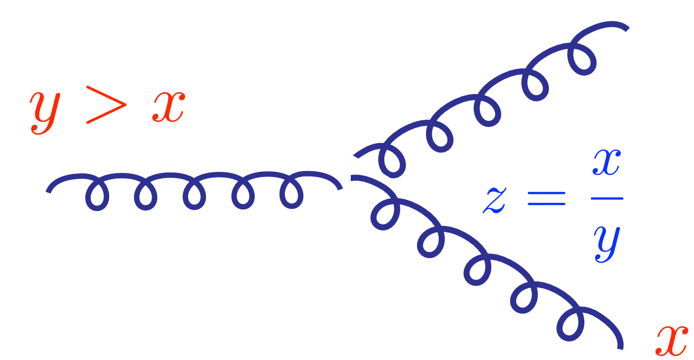
$$[P_{qq}(z)]_+ = C_F \left\{ \frac{1+z^2}{(1-z)_+} + \frac{3}{2}\delta(1-z) \right\}$$

  and could derive a similar result for $P_{gg}(z)$, again containing a δ-function term.

- These are called regularized splitting functions.
- They are often denoted simply by P_{ab}, with the unregularized forms denoted by a circumflex (beware my slight abuse of this notation in these lectures).
- The additional δ-function terms correspond to no momentum lost during the evolution.
  - they are interpreted as virtual corrections;
  - this formalism is thus often said to include part of the higher-order corrections.



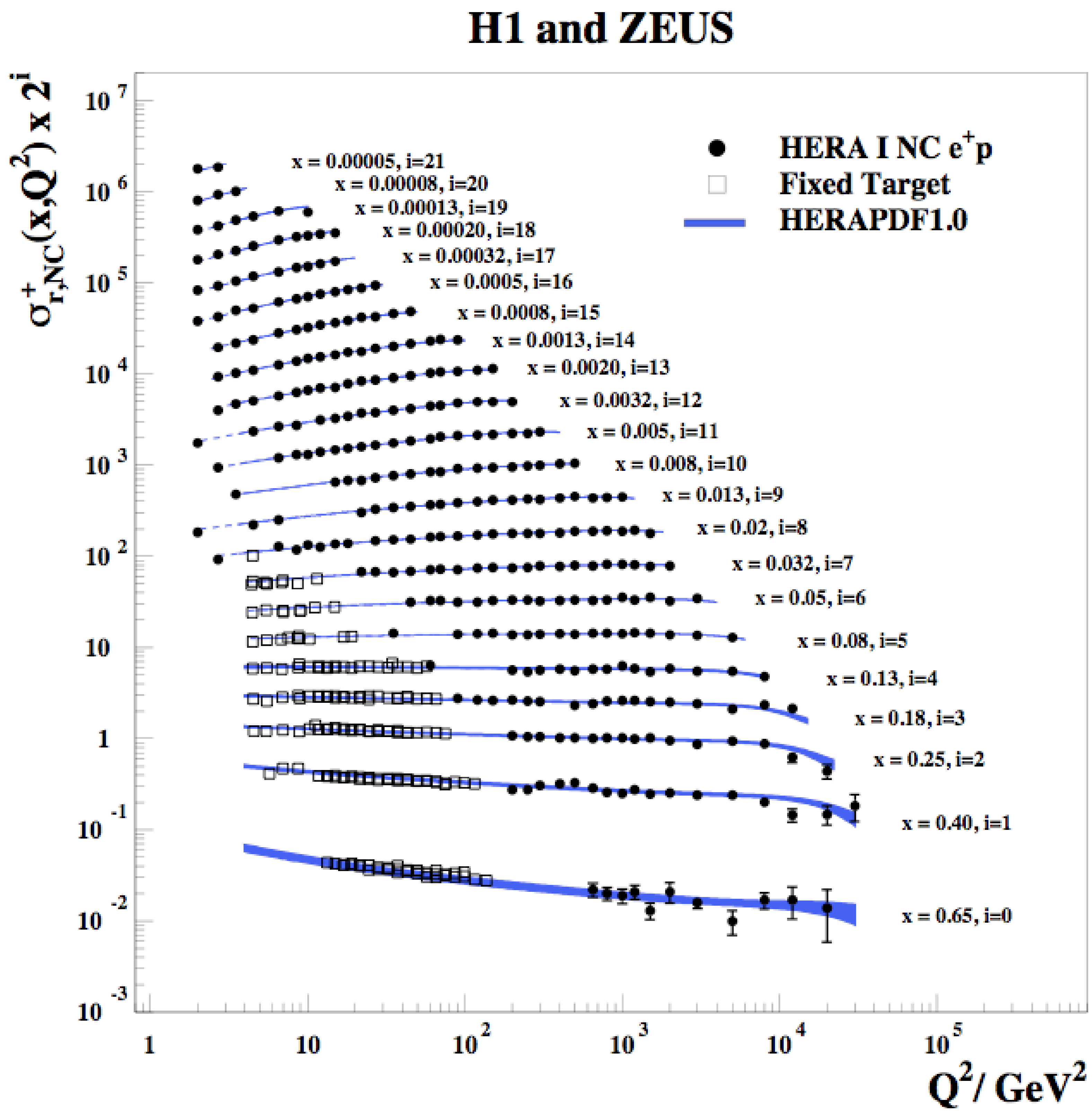
evolution by virtual emission and re-absorption ($z=1$)



evolution by branching ($z<1$)

Experimental confirmation

- Combination of HERA data (H1 and ZEUS experiments) over the period from 1994 to 2000.
- Scaling violation predicted in the QCD-improved parton model is clearly visible in data.
- Although pdfs are not calculable in perturbative QCD, their evolution is.



Factorization

- Just as we saw with the strong coupling, performing calculations in this formalism beyond the leading order gives us singular predictions.
- Once again we have to absorb the singularities into a redefined quantity - this time the pdf - in order to recover any predictive power.
  - this introduces a new factorization scale, μ_F.
- Once done, the pdfs are now universal (do not depend on the process) and, in principle, their evolution calculable at any order in perturbation theory.
- The final form of our factorization theorem is then:

$$\sigma_{AB} = \int dx_a dx_b \, f_{a/A}(x_a, \mu_F^2) \, f_{b/B}(x_b, \mu_F^2) \, \hat{\sigma}_{ab \to X}$$

- It is worthwhile to remember that this formula has only been proven for a handful of processes, certainly not for everything in which we are interested.
  - nevertheless the success of this approach, in confronting data with calculations within perturbative QCD, tells its own story.

Pdfs: general strategy

- Since the Q^2 evolution of the pdfs is known, we just have to determine their form at some particular value (typically, Q_0=1-2 GeV).
- Traditional strategy: make an ansatz $g_i(x)$ for each of the pdfs at this scale, with number of free parameters (~20 total). For example:

$$f_i(x, Q_0^2) = A_i x_i^a \left(1 + b_i \sqrt{x} + c_i x\right) (1-x)^{d_i}$$

- Perform a global fit to relevant data, using DGLAP equation to evolve pdfs to the appropriate scale first. Plenty of room for interpretation:
  - choice of input data sets (esp. in cases of conflict);
  - order of perturbation theory;
  - input parameterization and other theoretical prejudice.
- Global fitting industry: a number of groups have been performing and refining this procedure over the years. Most-used today: CTEQ and MSTW.
- Relative newcomers NNPDF with a slightly different approach: use neural network to remove form/parameter bias, clearer estimate of uncertainties.

Example input data: MSTW2008

Data set	$N_{\rm pts.}$
H1 MB 99 e^+p NC	8
H1 MB 97 e^+p NC	64
H1 low Q^2 96–97 e^+p NC	80
H1 high Q^2 98–99 e^-p NC	126
H1 high Q^2 99–00 e^+p NC	147
ZEUS SVX 95 e^+p NC	30
ZEUS 96–97 e^+p NC	144
ZEUS 98–99 e^-p NC	92
ZEUS 99–00 e^+p NC	90
H1 99–00 e^+p CC	28
ZEUS 99–00 e^+p CC	30
H1/ZEUS $e^\pm p$ $F_2^{\rm charm}$	83
H1 99–00 e^+p incl. jets	24
ZEUS 96–97 e^+p incl. jets	30
ZEUS 98–00 $e^\pm p$ incl. jets	30
DØ II $p\bar{p}$ incl. jets	110
CDF II $p\bar{p}$ incl. jets	76
CDF II $W \to l\nu$ asym.	22
DØ II $W \to l\nu$ asym.	10
DØ II Z rap.	28
CDF II Z rap.	29

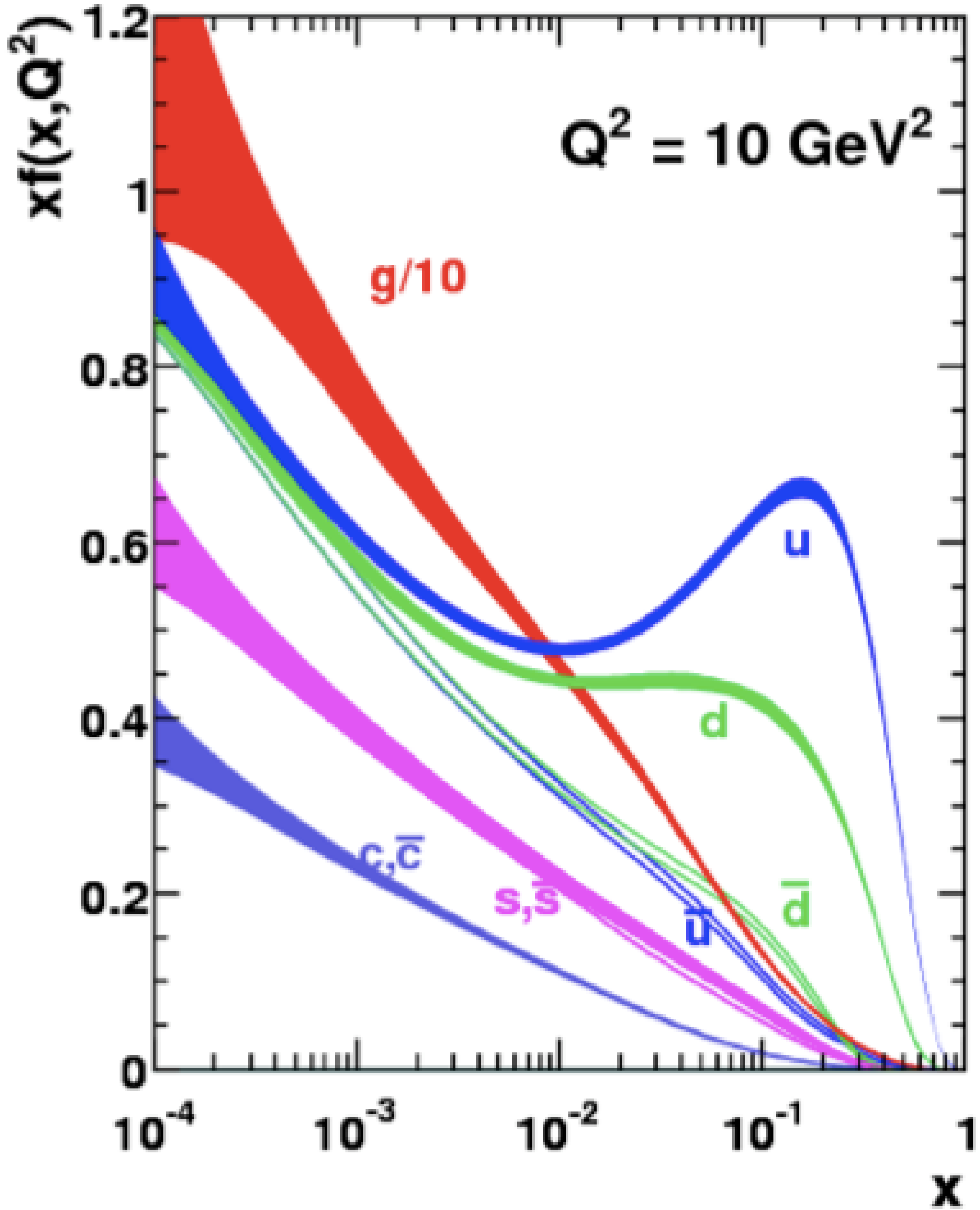
Data set	$N_{\rm pts.}$
BCDMS μp F_2	163
BCDMS μd F_2	151
NMC μp F_2	123
NMC μd F_2	123
NMC $\mu n/\mu p$	148
E665 μp F_2	53
E665 μd F_2	53
SLAC ep F_2	37
SLAC ed F_2	38
NMC/BCDMS/SLAC F_L	31
E866/NuSea pp DY	184
E866/NuSea pd/pp DY	15
NuTeV νN F_2	53
CHORUS νN F_2	42
NuTeV νN xF_3	45
CHORUS νN xF_3	33
CCFR $\nu N \to \mu\mu X$	86
NuTeV $\nu N \to \mu\mu X$	84
All data sets	**2743**

- Red = New w.r.t. MRST 2006 fit.

J. Stirling

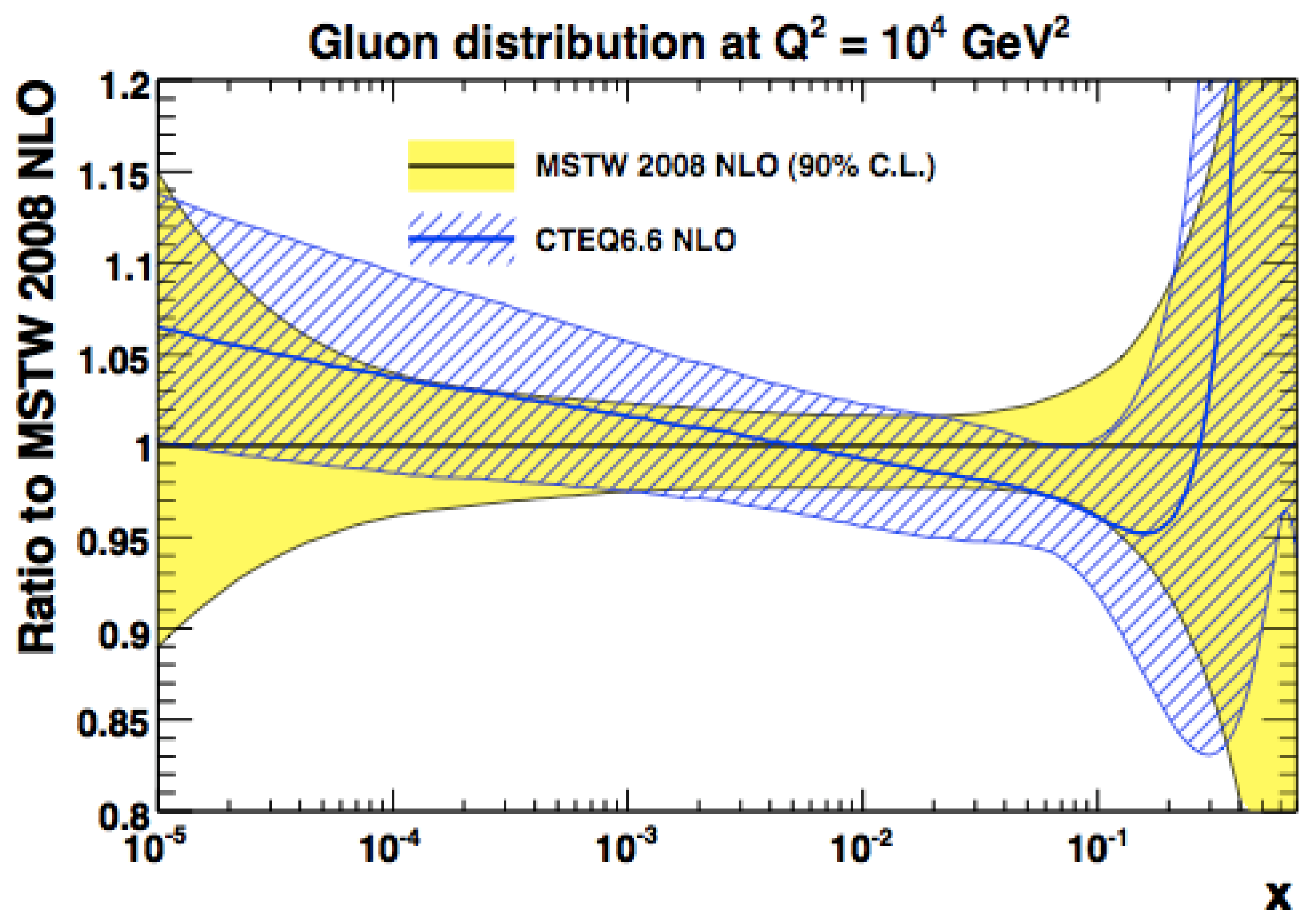
Example output

MSTW2008



gluons very important
at small x (LHC)

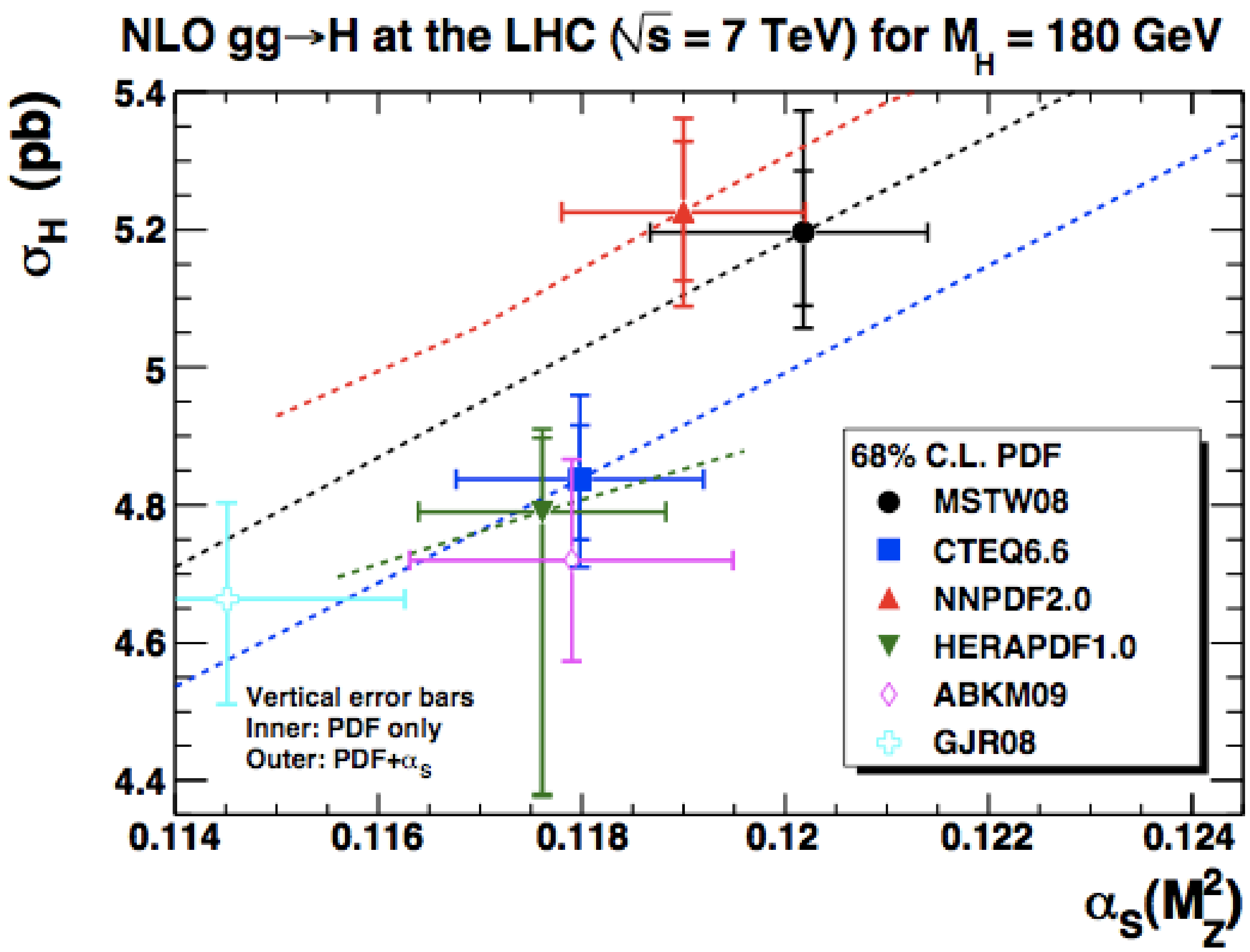
CTEQ6.6/MSTW2008 comparison



Broad agreement between the two different fits, but differences that become important when trying to make precision predictions.

PDF differences

- Cross section for a putative Higgs, produced through the gluon-fusion channel we discussed before, can be particularly sensitive to these differences.

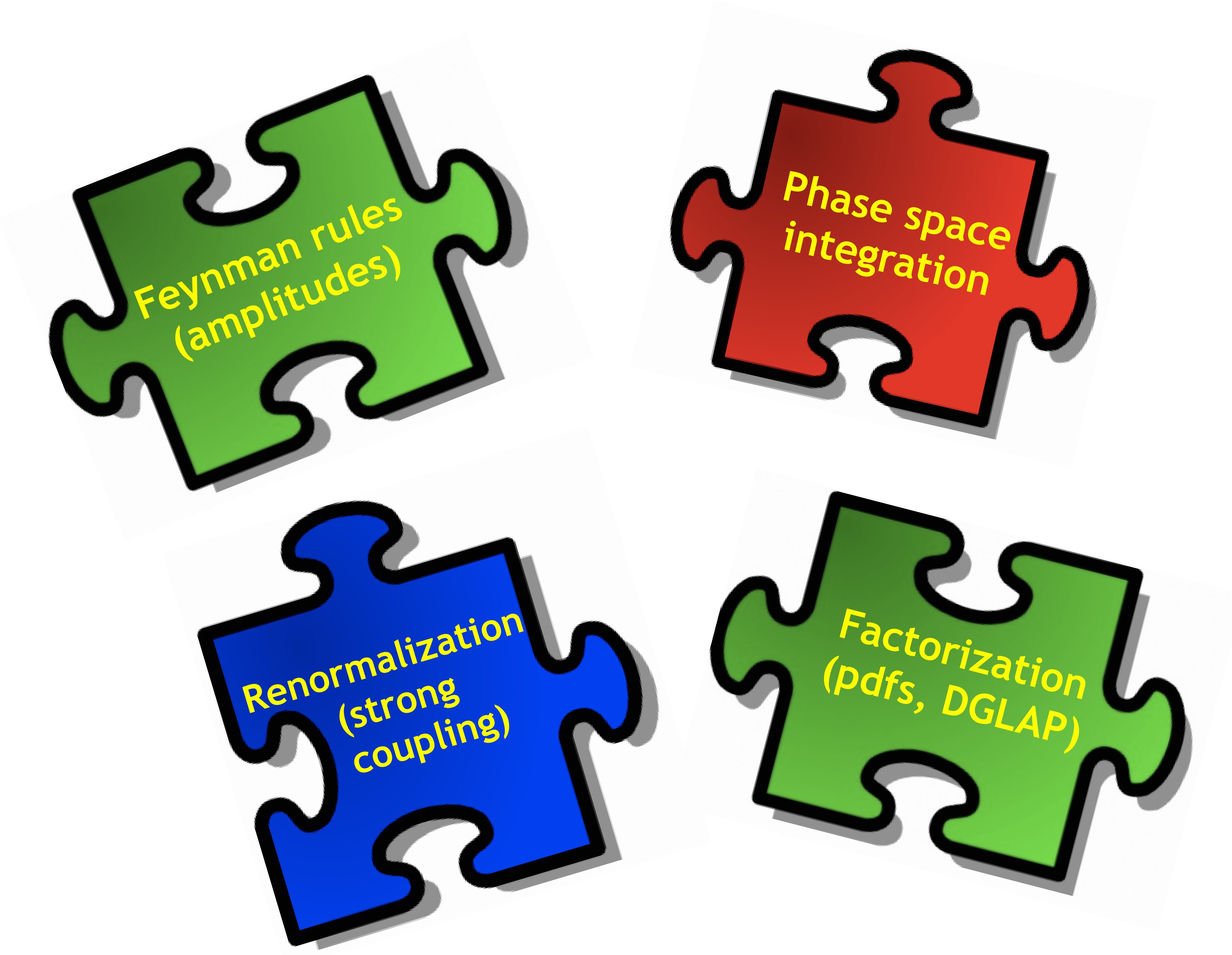


G. Watt
March 2010

Cross section variation ~ 10% from pdfs alone (mostly input α_S)

QCD phenomenology: ingredients

- We now have all the ingredients for QCD phenomenology at hadron colliders.



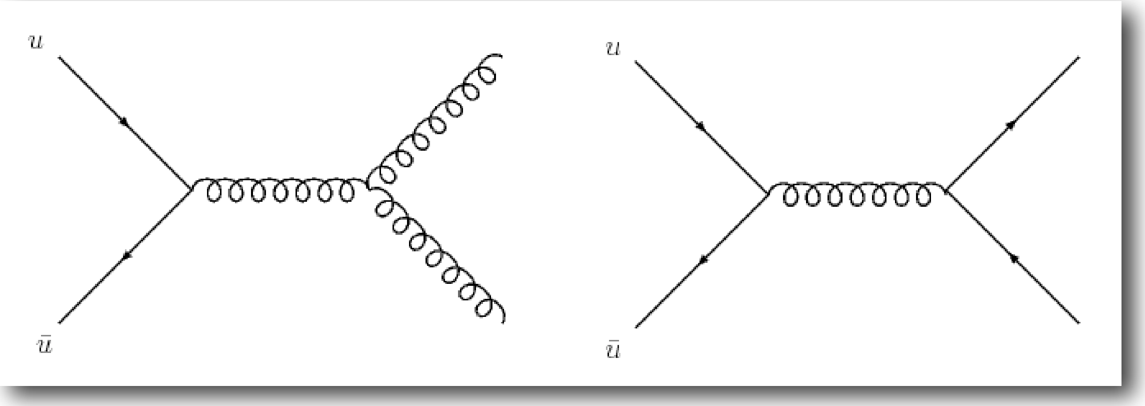
Price to pay: introduction of renormalization and factorization scales, μ_F and μ_R

Scale choices

- The two scales that we have introduced are artefacts of the perturbative approach: there is no dependence on them in the full theory.
- By truncating at a particular order in perturbation theory, we retain some dependence upon them.
  - formally, the QCD beta function (DGLAP equation) tells us the form of the renormalization (factorization) scale dependence we should expect;
  - in practice, the numerical effect of this dependence may not be small and varies with the particular calculation at hand.
- Often we argue that these arbitrary scales should be set equal to a typical mass scale in the process.
  - e.g. for inclusive W production, hard to argue with M_W.
  - in presence of additional hard radiation, answer is less clear (p_T(jet), Σp_T ...)
- Even when we are happy with a "typical scale" on kinematic grounds, one can always argue about the numerical coefficient in front of it.

Example

- Consider the single-jet inclusive distribution at the Tevatron.
  At high E_T (i.e. large x) it is dominated by the quark-antiquark initial state.



N. Glover, 2002

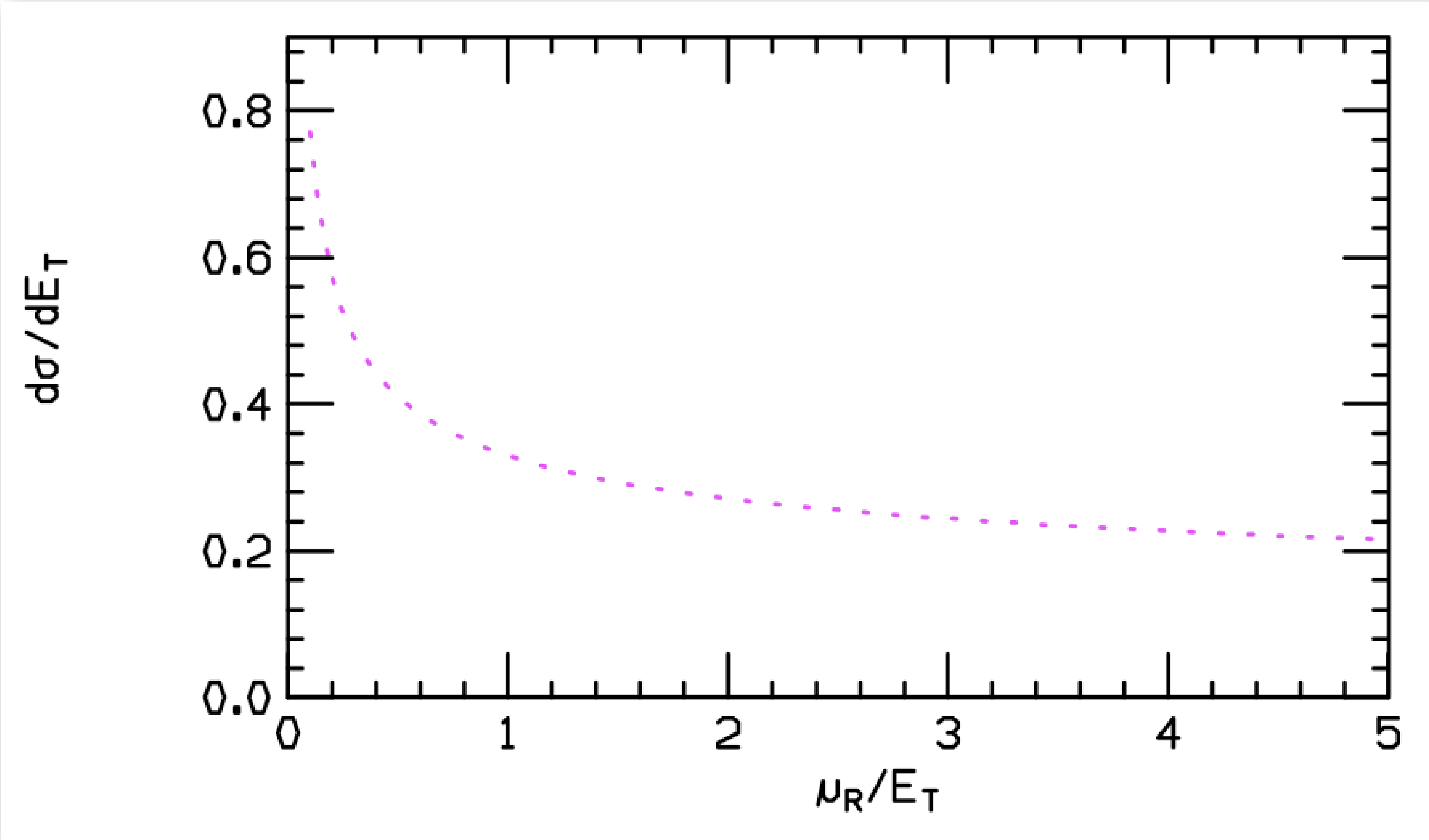
- We can write the result, up to next-to-leading order (NLO), as follows:

$$\frac{d\sigma}{dE_T} = \left[\alpha_s^2(\mu_R)\,\mathcal{A} + \alpha_s^3(\mu_R)\Big(\mathcal{B} + 2b_0\log(\mu_R/E_T)\,\mathcal{A} - 2P_{qq}\log(\mu_F/E_T)\,\mathcal{A}\Big)\right]$$
$$\otimes f_q(\mu_F) \otimes f_{\bar{q}}(\mu_F).$$

shorthand for convolution with PDF

- The leading order result, A, is proportional to α_s^2.
- At the next order, logarithms of the renormalization and factorization scales appear (c.f. renormalization discussion before) and are written explicitly here.
  - the remainder of the α_s^3 corrections lie in the function B.

Scale dependence: LO



- The distribution at the Tevatron, for *E_T=100 GeV*. The factorization scale is kept fixed at $\mu_F = E_T$ and the ratio μ_R/E_T varied about a central value of 1.
- At this order, the dependence just reflects the running of αs. The prediction varies considerably as μ_R is changed→ normalization of the cross section is unreliable. This is typical.

Scale dependence: NLO

$$\frac{d\sigma}{dE_T} = \Big[\alpha_s^2(\mu_R)\,\mathcal{A} + \alpha_s^3(\mu_R)\Big(\mathcal{B} + 2b_0\log(\mu_R/E_T)\,\mathcal{A} - 2P_{qq}\log(\mu_F/E_T)\,\mathcal{A}\Big)\Big]$$
$$\otimes f_q(\mu_F) \otimes f_{\bar{q}}(\mu_F).$$

- Now consider dependence of this observable on μ_F and μ_R at NLO.
- First recall the definition of the beta function and the DGLAP equation for f_q.

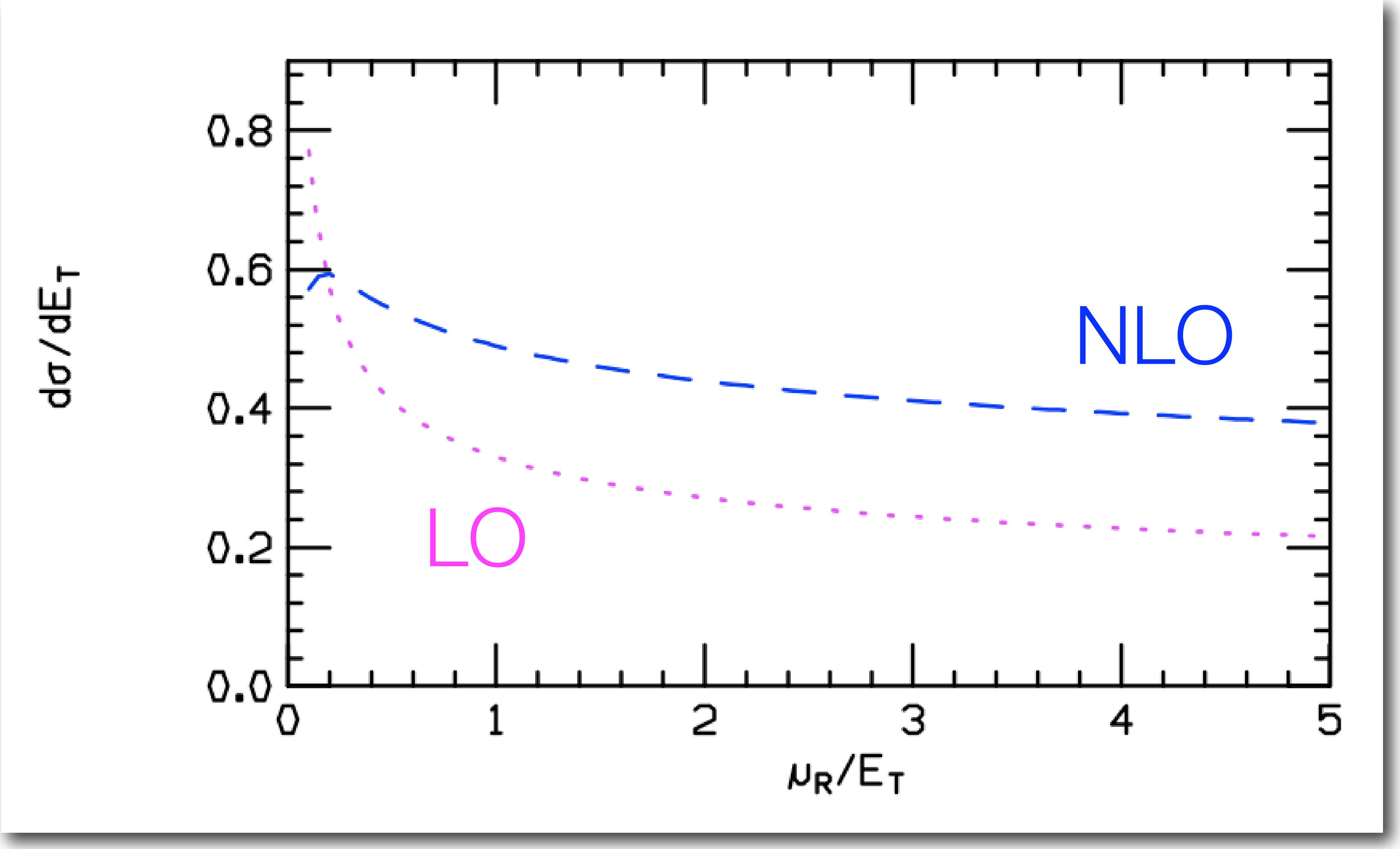
$$\beta(\alpha_s) = \mu\,\frac{\partial\alpha_s}{\partial\mu} \qquad \Longrightarrow \quad \frac{\partial\alpha_s(\mu_R)}{\partial\log\mu_R} = -b_0\alpha_s(\mu_R)^2 + \mathcal{O}(\alpha_s^3)$$

$$\frac{\partial f_i(\mu_F)}{\partial\log\mu_F} = \alpha_s(\mu_R)P_{qq}\otimes f_i(\mu_F) + \mathcal{O}(\alpha_s^2)$$

- We then see that: $\dfrac{\partial}{\partial\log\mu_R}\left[\dfrac{d\sigma}{dE_T}\right] = \mathcal{O}(\alpha_s^4)\,, \quad \dfrac{\partial}{\partial\log\mu_F}\left[\dfrac{d\sigma}{dE_T}\right] = \mathcal{O}(\alpha_s^4)$

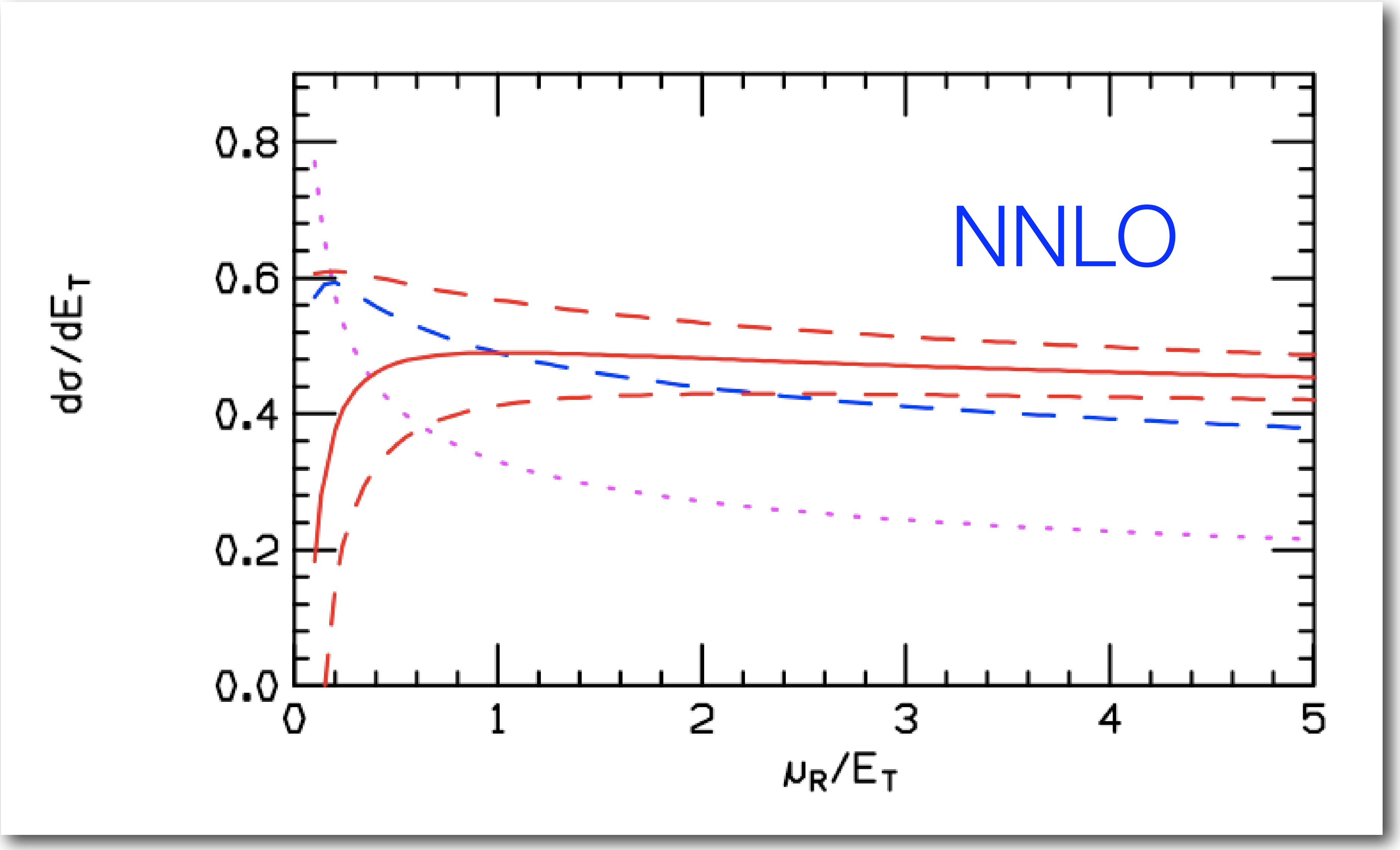
- In other words, the NLO result is explicitly independent of the renormalization and factorization scales, up to terms that are formally higher order.

Scale dependence: NLO



- At NLO, the growth as μ_R is decreased is softened by the logarithm that appears with coefficient α_s^3. Resulting turn-over is typical of NLO prediction.
- As a result, the range of predicted values at NLO is much reduced we obtain the first reliable normalization of the prediction.

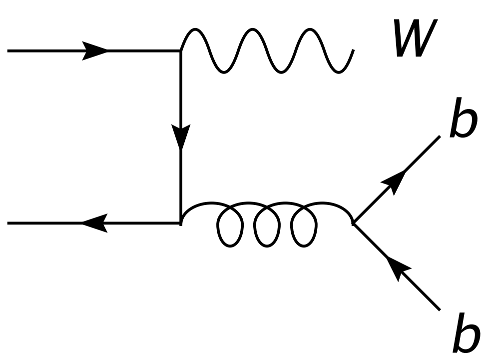
Scale dependence: NNLO



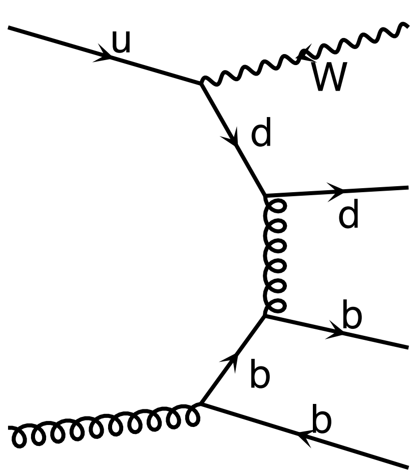
- The NNLO calculation for this process has not been completed, but one can see the effect of reasonable guesses for the single unknown coefficient.
  - would give a first reliable estimate of theoretical error, around a few percent $\rightarrow$ this is the level desired (required) for many LHC analyses.

A word of caution

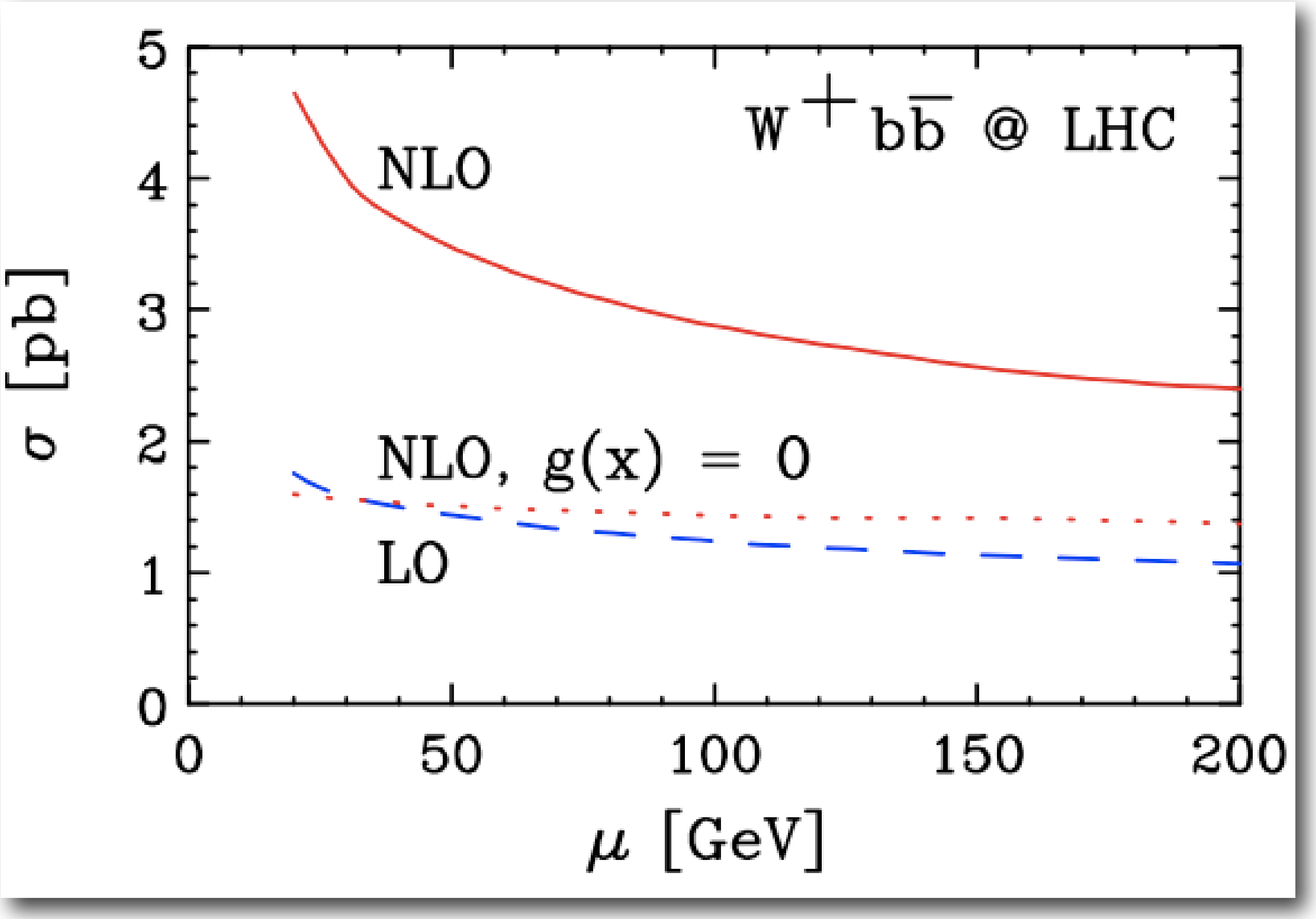
- These scale dependence plots are typical, but not universal.
- The LHC can provide plenty of counter-examples, due to the large gluon pdf.
- Case in point: Wbb production.



LO



NLO



Interpretation: expect plenty of jets when considering the Wbb final state at the LHC

Recap

- Virtual loops that appear beyond leading order contain ultraviolet singularities
  - these need to be regularized and then renormalized away, introducing dependence on an arbitrary renormalization scale, μ_R.
  - this procedure requires the strong coupling to run according to the beta function, which also determines dependence of an observable on μ_R.
- Parton distribution functions (pdfs) describe the partonic content of protons.
  - the simplest picture is modified by QCD, with pdfs being dependent upon a mass scale (at which the proton is probed) → DGLAP evolution again.
  - calculating beyond leading order we again find singularities that must be absorbed into a redefinition of the pdfs, introducing factorization scale μ_F.
  - a number of global pdf fits are available: consistent, but details differ.
- At hadron colliders, dependence on the new scales μ_R and μ_F can be large at leading order.
  - generic improvement at higher orders, but not guaranteed.