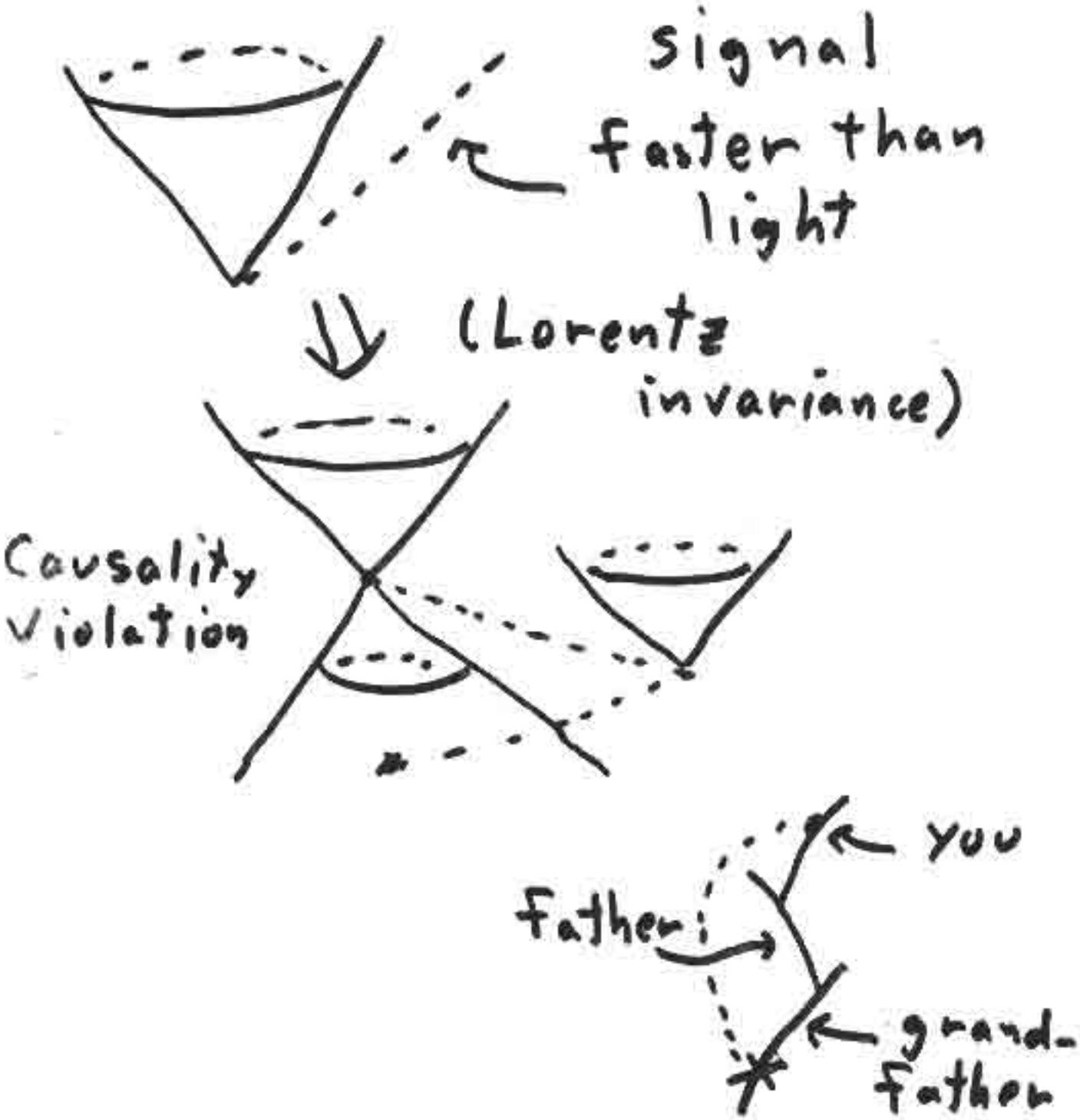
Grandfather Paradox



Symmetric - Hyperbolic Systems

M 4-dim (events)

Fields φ^α $\alpha = 1, \dots, n$

Eqns:

$$\mathcal{A}^a{}_{\alpha\beta}(x,\varphi)\nabla_a \varphi^\alpha = j_\beta(x,\varphi)$$

[1st order, quasilinear] [+ others "constraints"]

symmetric: $\mathcal{A}^a{}_{\alpha\beta} = \mathcal{A}^a{}_{\beta\alpha}$.

Causal cone C:

w^a: $w^a s_a > 0$ for all s_a with $s_a \mathcal{A}^a{}_{\alpha\beta}$ + def

"propagation directions"

Initial - Value Formulation

S not "cut" C (i.e. $n_a k^a{}_{\alpha\beta}$ + def)

S "initial time"

fields $\varphi^\alpha{}_0$ given on S (initial data)
(satisfy constr)

Then:

1. Exists unique solution φ^α, with $\varphi^\alpha|_S = \varphi^\alpha{}_0$.

2. φ^α here depends only on data here

Examples

Maxwell: $\varphi^\alpha = F_{ab}$

$\nabla^b F_{ab} = 0 \quad \nabla_{[a} F_{bc]} = 0$

C = light cone

Wave Eqn $\Box \varphi = 0$

$\varphi^\alpha = (\varphi, v_a)$

$\nabla_a \varphi = v_a, \quad \nabla_{[a} v_{b]} = 0 \quad \nabla^a v_a = 0$

C = light cone

Also, Klein-Gordon $\Box\varphi - m^2\varphi = 0$.

Also Tachyon $\Box\varphi + m^2\varphi = 0$!

Examples (Cont)

General Relativity

$$\phi^\alpha = (\underline{g_{ab}}, \underline{\nabla_a})$$

metric deriv op

$$\nabla_a \nabla_b \xi_c - \nabla_b \nabla_a \xi_c = \frac{1}{2} R_{abc}{}^d \xi_d$$

$R_{abc}{}^d$ Riemann tensor = "$\nabla(\nabla)$".

Eqns:

$$\underline{\nabla_a} g_{bc} = 0$$

$$\underline{R_{mbc}{}^m = 0}$$

System is symmetric-hyperb (ignoring some "gauge freedom")

C = light cone of g_{ab}.

Examples (Cont)

$\frac{dp}{d\rho} = v^2_{\text{sound speed}}$

Fluid φ^α:

u^a (4-velocity : unit timelike)

ρ (mass density : >0)

$p(\rho)$ given (function of state)

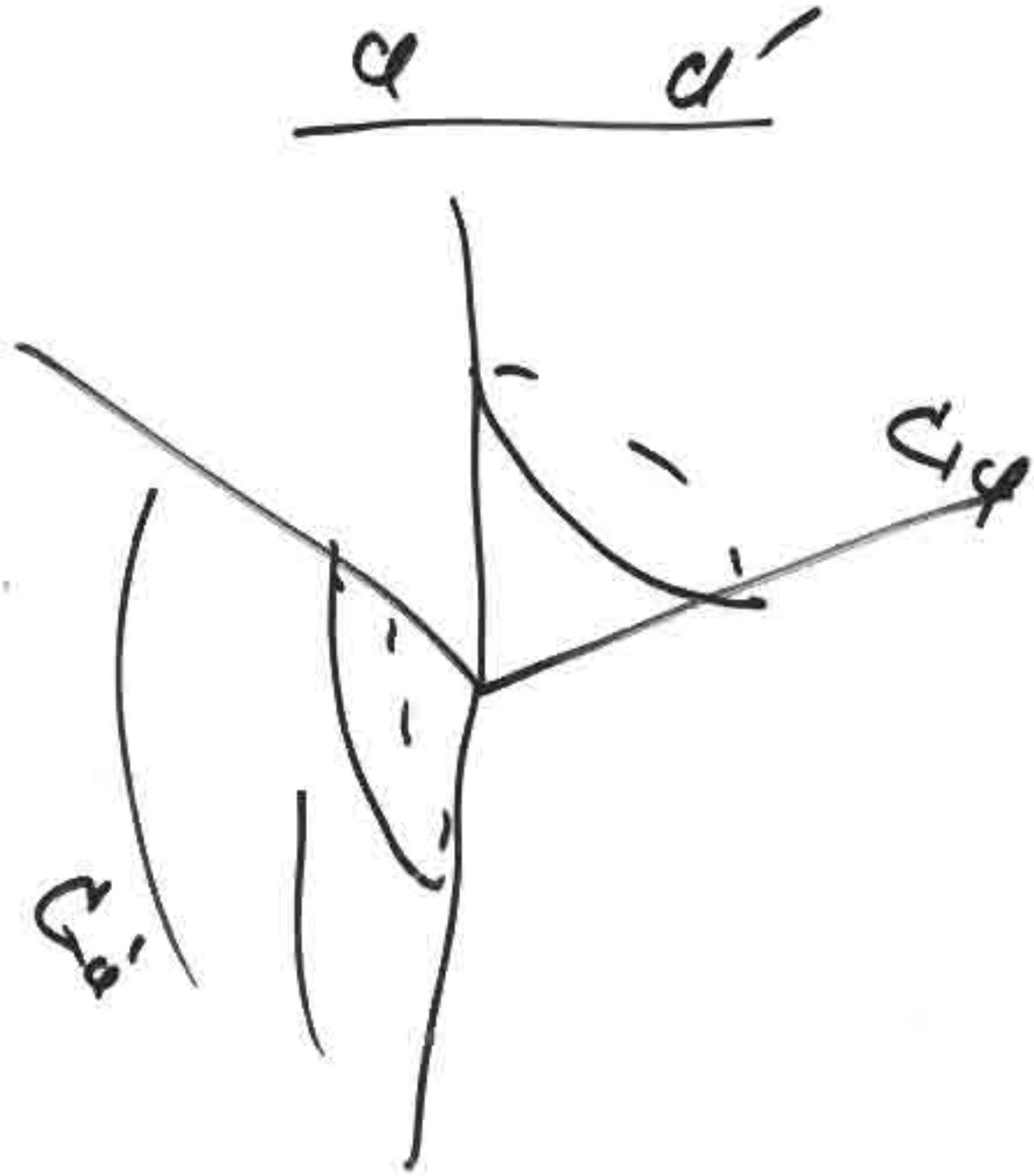
Eqns:

$$(\rho + p)\, u^m \nabla_m u^a = -(g^{ab} + u^a u^b)\, \nabla_b p$$

$$\nabla_a(\rho\, u^a) = -p\,(\nabla_a u^a)$$

$C = \{ w^a \mid \text{speed}^2 \text{ (relative to } u) \le \frac{dp}{d\rho} \}$

"sound cone"


a
a'
C_φ
$C_{\varphi'}$

Combining Systems

✓ φ $\qquad \lambda(x,\varphi)\,\nabla\varphi = \underline{j(x,\varphi)}$

✓ φ' $\qquad \lambda'(x,\varphi')\,\nabla\varphi' = \underline{j'(x,\varphi')}$

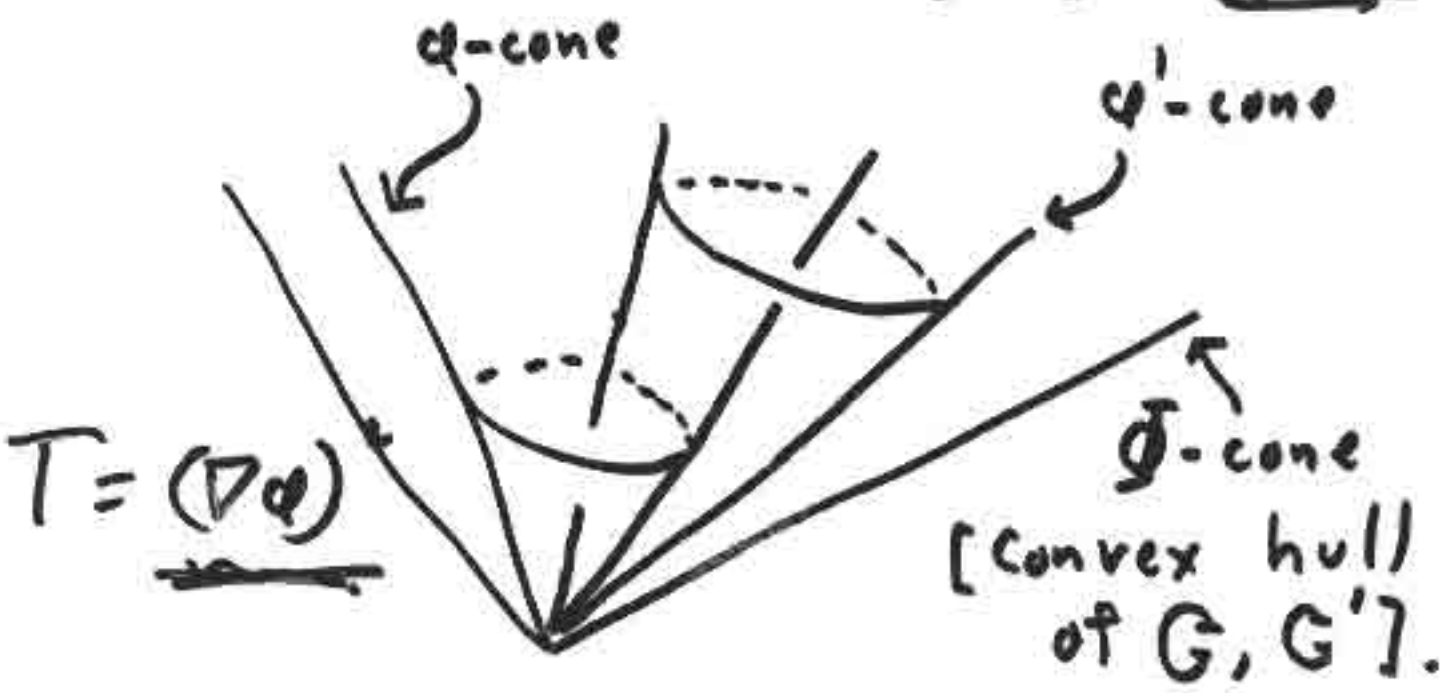
Symmetric-hyperbolic

$\underline{\Phi = (\varphi, \varphi')}$ $\qquad K = (\lambda, \lambda')$

$\underline{K(x,\Phi)\,\nabla\Phi = J(x,\Phi)}$

Also symm-hyperbolic

$\hookrightarrow j(x,\varphi) + j'(x,\varphi') + \underline{\text{interact}(x,\varphi,\varphi')}$



Gen Rel + Fluid

$$\phi^\alpha = (g_{ab}, \nabla, u^a, \rho)$$

$$\nabla_a g_{bc} = 0$$

$$R_{amb}{}^m = (\rho + p) u_a u_b + p\, g_{ab}$$

$$(\rho + p) u^m \nabla_m u^a = -(g^{ab} + u^a u^b) \nabla_b p$$

$$\nabla_a (\rho u^a) = -p\, \nabla_a u^a$$

C = larger of light cone and sound cone

$$\frac{dp}{d\rho} < c^2 \qquad \frac{dp}{d\rho} > c^2$$

$\Rightarrow$ exist soln, causal!

Initial-Value Formulation of Special Relativity!

$\varphi^\alpha = (g_{ab}, \nabla)$ $\quad U, \rho$

$$\left.\begin{aligned} \nabla_a g_{bc} &= 0 \\ R_{abc}{}^d &= 0 \end{aligned}\right\} \text{quasilinear, } 1^{st} \text{ order}$$

$R_{abc}{}^d = 0 \iff R_{mbc}{}^m = 0$ (Einstein) + some constraints

C = light cone

satisfy constraints

$\Rightarrow$ solution (spacetime of Special relativity)

$$\begin{cases} k(\underline{h}) \nabla F \\ k(\underline{g}) \nabla g \\ k(h) \nabla \underline{h} \end{cases}$$