# Highway Car Traffic as a Complex System

## The Physicist's Point of view

Nino Boccara

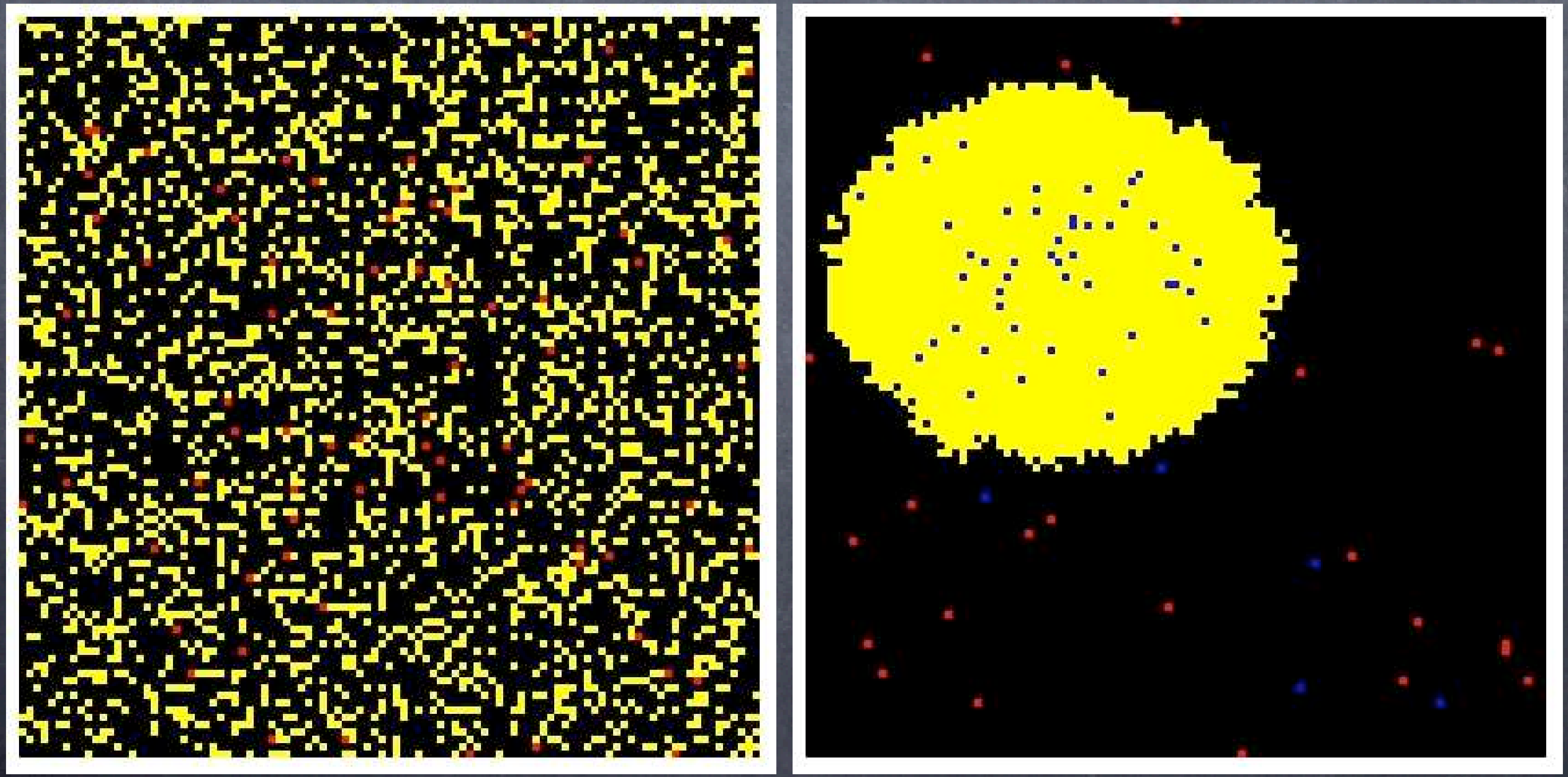
# What is a complex system?

- it consists of a large number of interacting agents,
- it exhibits emergence, that is, a self-organizing collective behavior difficult to anticipate from the knowledge of agents behavior,
- its emergent behavior does not result from the existence of a central controller.

N. Boccara, *Modeling Complex Systems*,
(New York: Springer-Verlag 2004)

Example: Project inspired by the behavior of termites gathering wood chips into piles.
http://education.mit.edu/starlogo/

Randomly distributed wood chips (left figure) eventually end up in a single pile (right figure). Density of wood: 0.25, number of termites: 75.



# What is a model?

- A model is a simplified mathematical representation of a system.
- In the actual system many features are likely to be important.
- Not all of them, however, should be included in the model.
- Only a few relevant features which are thought to play an essential role in the interpretation of the observed phenomena should be retained.
- A simple model, if it captures the key elements of a complex system, may elicit highly relevant questions.

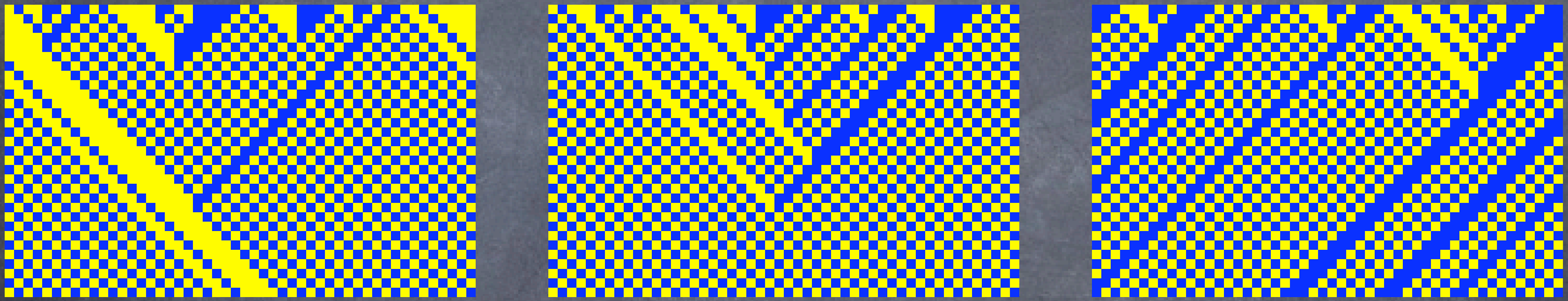
# Cellular Automaton Models of Highway Traffic Flow

- First model: K. Nagel and M. Schreckenberg:
  *A Cellular Automaton Model for Freeway Traffic,*
  Journal de Physique I **2** 2221-2229 (1992)

- Review: D. Chowdhury, L. Santen, and A. Schadschneider,
  *Statistical Physics of VehicularTraffic and Some Related Systems,*
  Physics Reports **329** 199-329 (2000)

# Simplest model:
# A one-lane highway traffic flow

A linear array of cells either occupied (by a car) or empty



free-moving phase  critical state  jammed phase

At each time step a car (a blue square) either moves to the neighboring right cell if, and only if, it is empty (a yellow square) or does not move.

Each figure represents 30 time steps for different car densities.

## This very simple model shows:

- The existence of two phases: a free-moving phase for a car density less than a critical value and a jammed phase for a car density greater than this critical value.
- Local jams (sequence of stopped cars) are moving backwards.

This model is formulated in terms ofcellular automata.
A cellular automaton is a dynamical system

# A slightly more general model

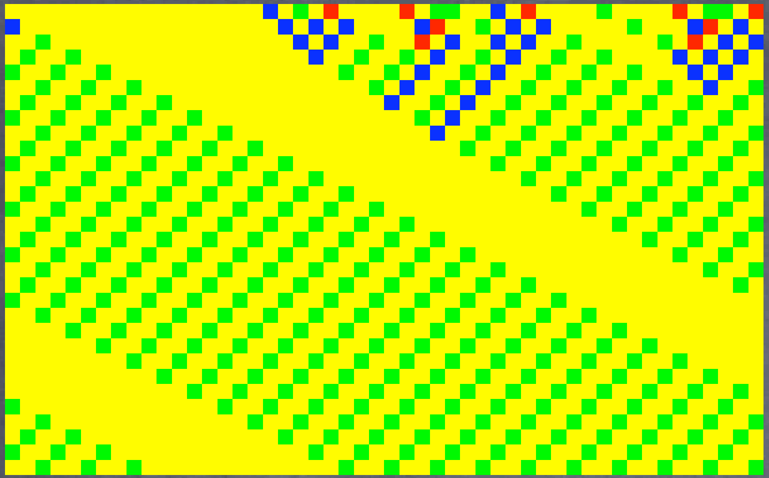
The previous model is a simplified deterministic version of the Nagel-Schreckenberg model.
It assumes that
1- car speeds can only take two values: 0 or 1, and
2- car drivers always move if they can

If we assume that car speeds take the values

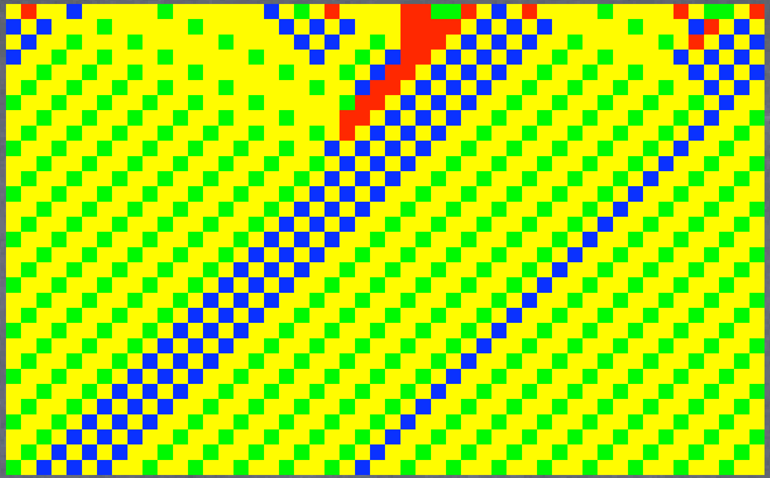
$$0, 1, 2, \ldots, v_{\max}$$

and adopt the moving rule: a car occupying cell i and having d empty cells on its right moves, at the next time step, to cell

$$j = i + \min(d, v_{\max})$$



free-moving phase



jammed phase

yellow: empty cell, red: speed 0, blue: speed 1, green: speed 2
color code at time t indicates speed at time t-1

critical density given by:

$$\rho_C = \frac{1}{v_{\max +1}}$$

Deterministic cellular automaton models of highway traffic flow obey a variational principle which states that, for a given car density, the average car speed is a non-decreasing function of time.

N. Boccara, *On the existence of a variational principle for deterministic cellular automaton models of highway traffic flow*,
International Journal of Modern Physics C **12** 1-16 (2001)

# What is a dynamical system?

The notion of dynamical system includes the following ingredients:

- A phase space $S$ whose elements represent possible states of the system;
- time $t$, which may be discrete or continuous;
- and an evolution law, that is, a rule that allows to determine the state at time $t$ from the knowledge of the states at all previous times.

# Two examples of dynamical systems:

## 1- Bulgarian solitaire

A pack of N = n (n+1)/2 cards is divided into k packs of

$$n_1, n_2, \ldots, n_k$$

cards, where

$$n_1 + n_2 + \cdots + n_k = N$$

A move consists in taking exactly one card of each pack and forming a new pack.

By repeating this operation a sufficiently large number of times any initial configuration eventually converges to a

configuration that consists of n packs of, respectively,
1, 2, . . ., n cards.
For instance, if N = 10 (which corresponds to n = 4), starting from the partition {1,2,7}, we obtain the following sequence:

{1,3,6}, {2,3,5}, {1,2,3,4}.

Numbers N of the form $n(n+1)/2$ are known as triangular numbers.

What happens if the number of cards is not triangular?

Since the number of partitions of a finite integer is finite, any initial partition leads into a cycle of partitions. For example, if $N = 8$, starting from {8}, we obtain the sequence:

{7, 1}, {6, 2}, {5, 2, 1}, {4, 3, 1 }, {3, 3, 2}, {3, 2, 2, 1}, {4,2,1,1}, {4,3,1}.

For any positive integer N, the convergence towards a cycle, which is of length 1 if N is triangular, has been proved by J. Brandt, *Cycles of Partitions*, Proceedings of the American Mathematical Society **85** 483--487 (1982).

In the case of a triangular number, it has been shown that the number of moves, before the final configuration is reached, is at most equal to $n(n-1)$.

The Bulgarian solitaire is a time-discrete dynamical system. Its phase space consists of all the partitions of the number N.

# 2- Original Collatz problem

Many iteration problems are simple to state but often intractably hard to solve. Probably the most famous one is the so-called 3x+1 problem, also known as the Collatz conjecture,which asserts that, starting from any positive integer $n$, repeated iteration of the function f defined by

$$f(n) = \begin{cases} \frac{1}{2}n, & \text{if } n \text{ is even,} \\ \frac{1}{2}(3n+1), & \text{if } n \text{ is odd,} \end{cases}$$

always returns 1.

Here is a less known conjecture that, like the 3x+1 problem, has not been solved. Consider the function f defined, for all positive integers, by

$$f(n) = \begin{cases} \frac{2}{3}n, & \text{if } n = 0 \mod 3, \\ \frac{4}{3}n - \frac{1}{3}, & \text{if } n = 1 \mod 3, \\ \frac{4}{3}n + \frac{1}{3}, & \text{if } n = 2 \mod 3. \end{cases}$$

Its inverse is defined by

$$f^{-1}(n) = \begin{cases} \frac{3}{2}n, & \text{if } n = 0 \mod 2, \\ \frac{3}{4}n + \frac{1}{4}, & \text{if } n = 1 \mod 4, \\ \frac{3}{4}n - \frac{1}{4}, & \text{if } n = 3 \mod 4. \end{cases}$$

The function f is bijective and is, therefore, a permutation of the natural numbers. The study of the iterates of f has been called the original Collatz problem

(see L. C. Lagarias, *The 3x+1 Problem and Its Generalizations*, American Mathematical Monthly **92** 3-23 (1985))

If we consider the first natural numbers, we obtain the following permutation:

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 1 | 3 | 2 | 5 | 7 | 4 | 9 | 11 | 6 |

While some cycles are finite, e.g., (3,2)
or (5,7,9,6,4), it has been conjectured that there exist infinite cycles. For instance, none of the 250,000 successive iterates of 8 is equal to 8. This is also the case for 14 and 16. For this particular dynamical system, the phase space is the set of all positive integers, and the evolution rule is reversible.

# What is a 1D cellular automaton?

$s(i,t) \in Q$ represents state at site $i \in \mathbb{Z}$ and time $t \in \mathbb{N}$

$$Q = \{0, 1, \ldots, q-1\}$$

the local evolution rule is a map

$$f : Q^{r_\ell + r_r + 1} \mapsto Q$$

such that

$$s(i, t+1) = f\big(s(i - r_\ell, t), \ldots, s(i + r_r, t)\big),$$

where

$$r_\ell \text{ and } r_r$$

are, respectively, the left and right radii of rule f.

# Critical behavior of a cellular automaton highway traffic model

N. Boccara and H. Fukś,
*Critical Behavior of a Cellular Automaton Highway Traffic Model* ,
Journal of Physics A: Mathematical and General **33** 3407-3415 (2000)

In the deterministic Nagel-Schreckenberg model the transition from the free-moving phase to the jammed phase may be viewed as second-order phase transition

1- What is the order parameter?
2- What is the symmetry-breaking field?
3- What is the physical quantity characterizing the linear response of the order parameter to the symmetry-breaking field?
4- What are the critical exponents?
5- Can we find scaling relations?

# Symmetry considerations

In a standard second-order phase transition, at high temperature, the system is in the disordered phase, i.e., the phase of higher symmetry.

Below a critical temperature, the system is in the ordered phase characterized by a nonzero value of a symmetry-breaking order parameter.

The symmetry group of the ordered phase is a subgroup of the symmetry group of the disordered phase.

For the transition from the free-moving phase to the jammed phase, the control parameter is the car density.

The average car velocity is exactly given by:

$$< v >= \min \left( v_{\max}, \frac{1}{\rho} - 1 \right)$$

This expression shows that below the critical car density:

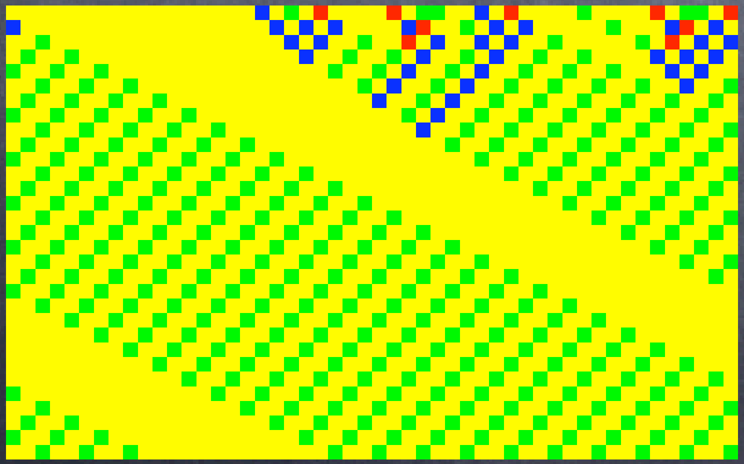
$$\rho_c = \frac{1}{v_{\max} + 1}$$

all cars move at the maximum velocity, while above the critical density, the average velocity is less than the maximum speed.

Cellular automata modeling one-lane traffic flow are number-conserving.

N. Boccara and H. Fukś,
*Number-conserving Cellular Automaton Rules* ,
Fundamenta Informaticae **52** 1-13 (2002)

Limit sets of number-conserving cellular automaton rules have, in most cases, a very simple structure, and are reached after a number of time steps of the order of the lattice size.



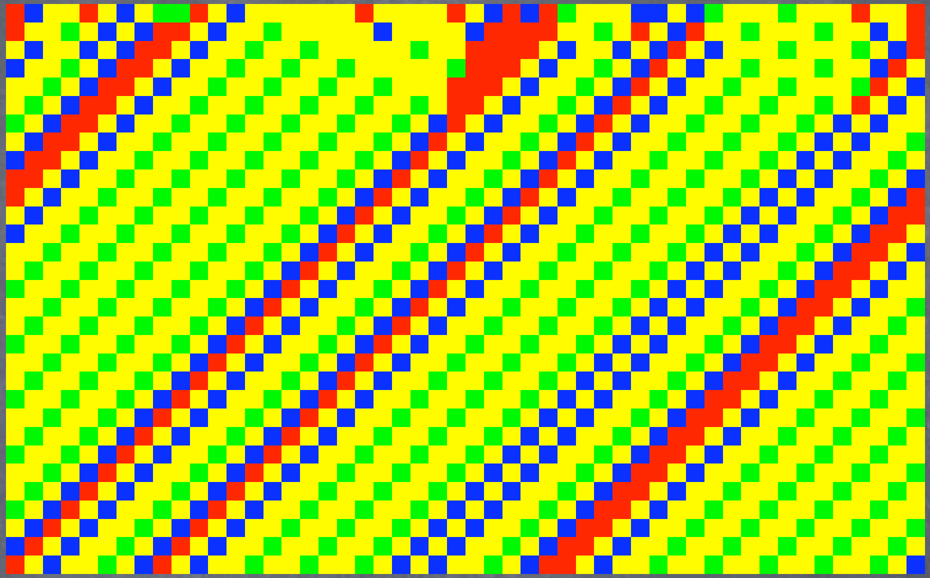
Below the critical density any configuration in the limit set consists of "perfect tiles" as shown below:

(e, e, 2)

in a sea of cells in state (e)

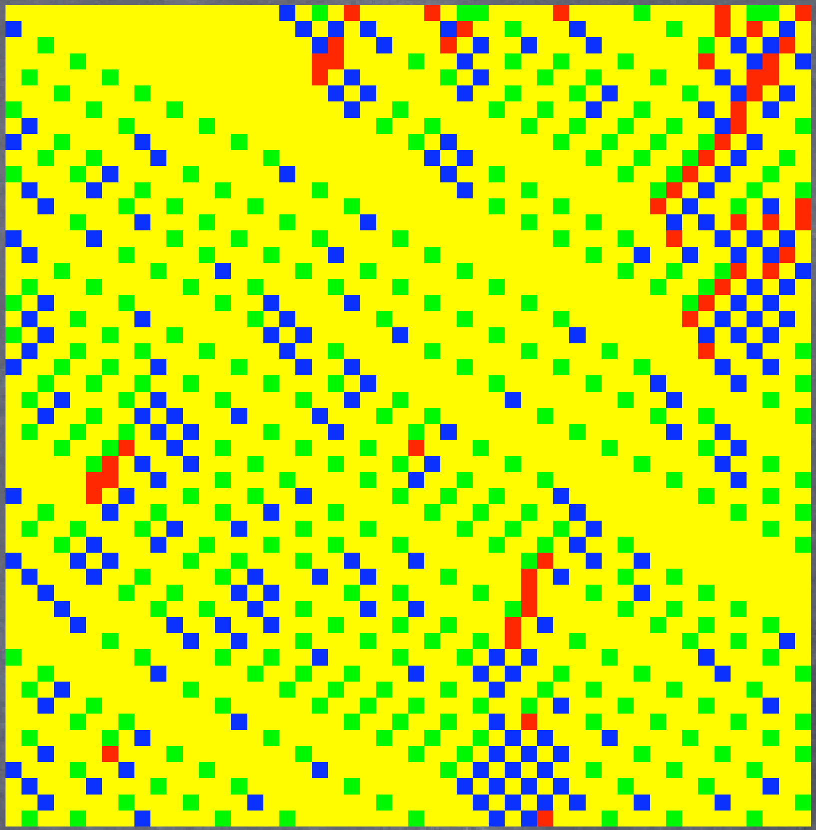
Above the critical density a configuration in the limit set consists of a mixture of tiles containing cells of type:

(e) (0) (e,1) (e,e,2)



Tiles of type (0) and (e,1) found in the limit set of the jammed phase are called defective

If in the deterministic model we introduce random braking, then, even at low density, some tiles become defective which causes the average velocity to be less than the maximum speed



random braking means that a driver who could move at velocity $v$ has a probability $p$ to move at velocity $v-1$

maximum speed: 2
car density: 0.2
braking probability: 0.25

The random braking parameter p can, therefore, be viewed as the symmetry-breaking field, and the order parameter m, conjugate to that field, is

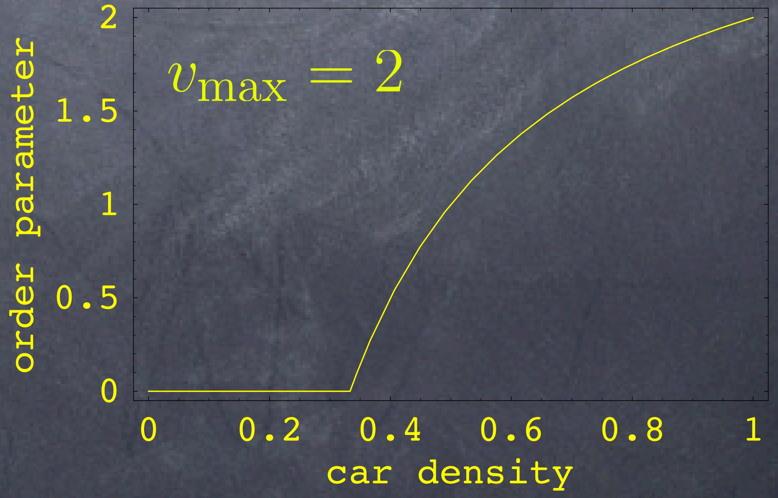
$$m = v_{\max} - < v >$$

since

$$< v > = \min\left(v_{\max}, \frac{1}{\rho} - 1\right)$$

we obtain

$$m = \begin{cases} 0, & \text{if } \rho < \rho_c, \\ \frac{\rho - \rho_c}{\rho \rho_c}, & \text{otherwise.} \end{cases}$$



# Critical exponents

For this particular system, in the vicinity of the transition point, the asymptotic behavior is characterized by the exponents $\beta$, $\gamma$, $\gamma'$, and $\delta$ defined by

$$m \sim (\rho - \rho_c)^{\beta}$$

$\beta$ is exactly equal to 1

$$\lim_{p\to 0} \frac{\partial m}{\partial p} \sim \begin{cases} (\rho_c - \rho)^{\gamma}, & \text{if } \rho < \rho_c, \\ (\rho - \rho_c)^{\gamma'}, & \text{if } \rho > \rho_c, \end{cases}$$

$$\lim_{p\to 0} \left(\frac{\partial m}{\partial p}\right)_{\rho=\rho_c} \sim p^{1/\delta}$$

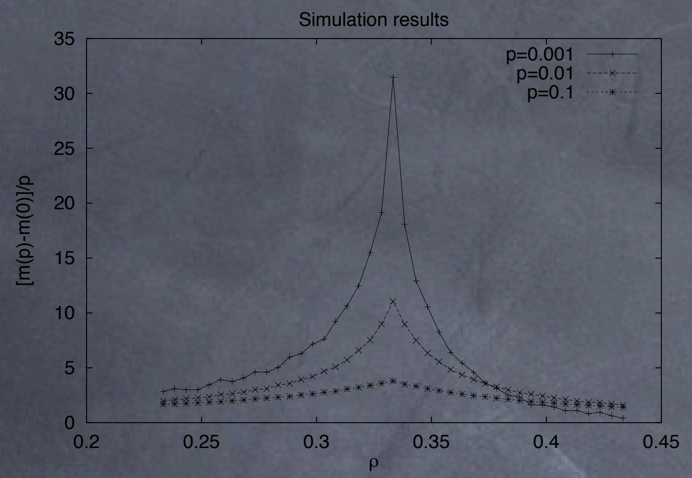
Close to the critical point, critical exponents satisfy scaling relations. If we assume that $m$ is a generalized homogeneous function of $p$ and $\rho-\rho_c$, that is,

$$m = |\rho - \rho_c|^{\beta} f\left(\frac{p}{|\rho - \rho_c|^{\beta\gamma}}\right)$$

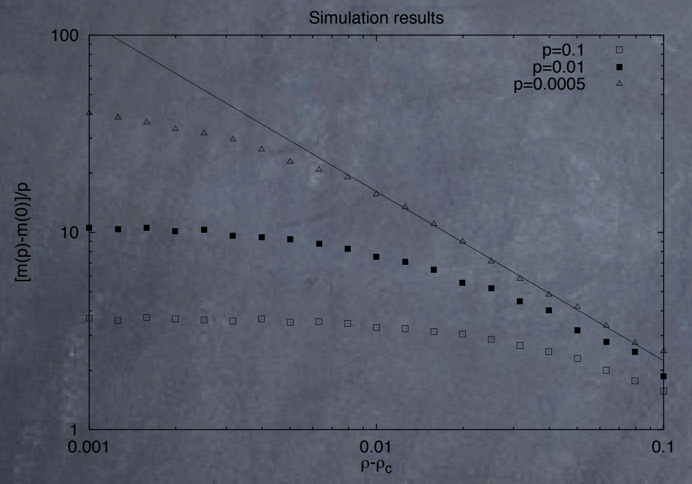
then

$$\gamma' = \gamma = (\delta - 1)\beta$$

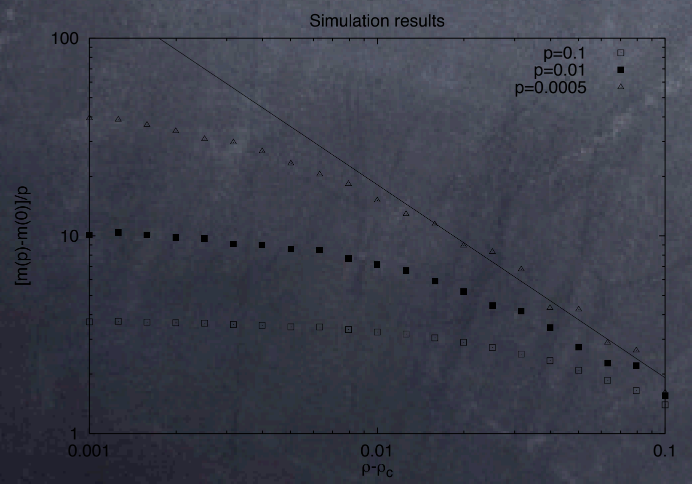
# Numerical simulations



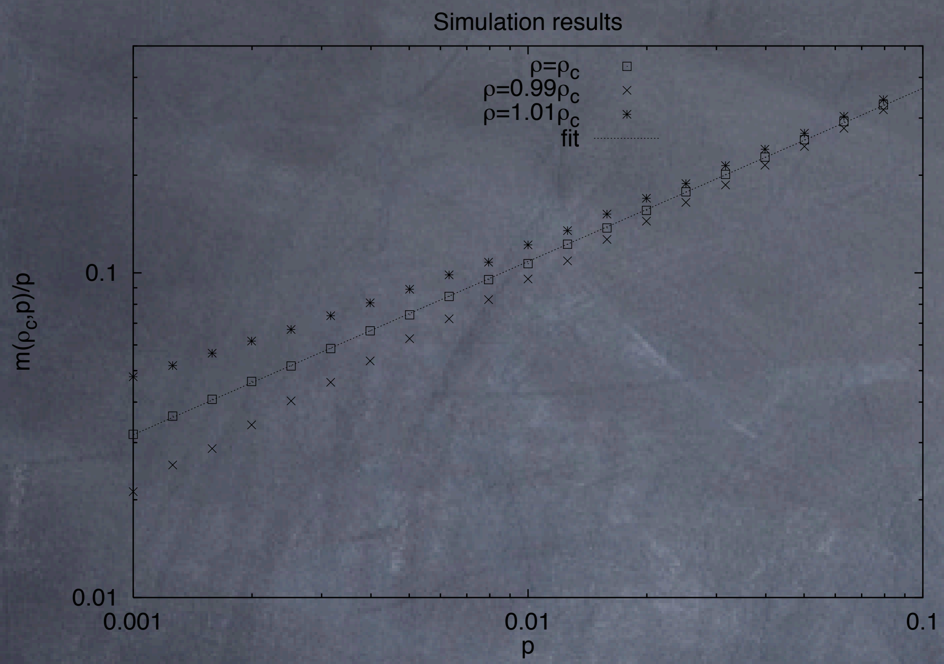

Divergence of $\partial m/\partial p$ at the critical point




Log-Log plot of $(m(p)-m(0))/p$ as a function of $|\rho-\rho_c|$ for $\rho<\rho_c$




Log-Log plot of $(m(p)-m(0))/p$ as a function of $|\rho-\rho_c|$ for $\rho>\rho_c$




Log-Log plot of m as a function of p for $\rho = \rho_c$

Numerical simulations show that

$\gamma \simeq \gamma' \simeq 1$ and $\delta \simeq 2$

These critical exponents satisfy the scaling relation $\gamma = (\delta - 1)\beta$

These results can also be obtained using an approximate technique, called local structure theory, at order 3 going beyond the mean-field approximation (order 0).

H. A. Gutowitz H. A., J. D. Victor J. D., and B. W. Knight
*Local Structure Theory for CellularAutomata*,
Physica D 28 18--48 (1987)

# THE END