The Million-Body Problem: Particle Simulations in Astrophysics

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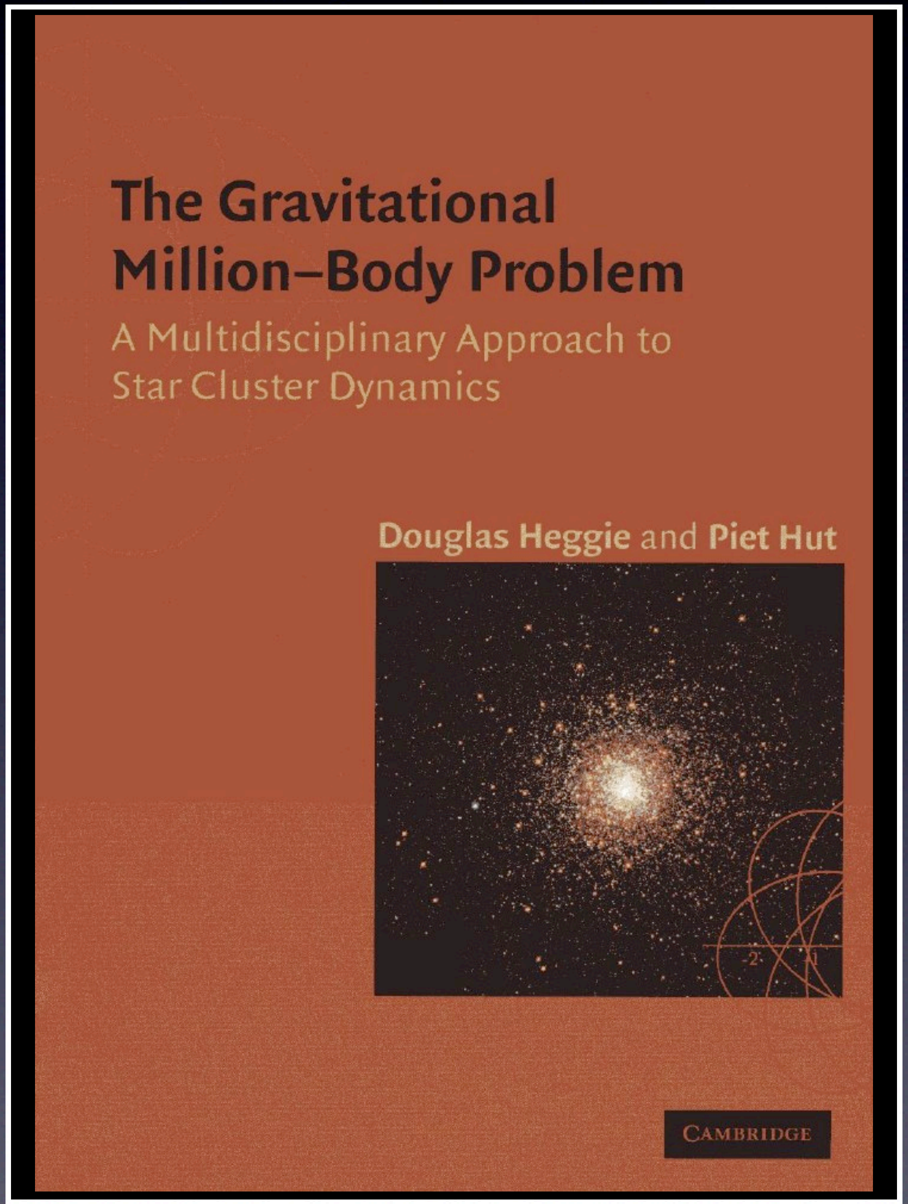
N-body problems

- Ubiquitous in astrophysics
- N "particles" coupled through gravity
- N is large (10^5 - 10^7 is typical nowadays)

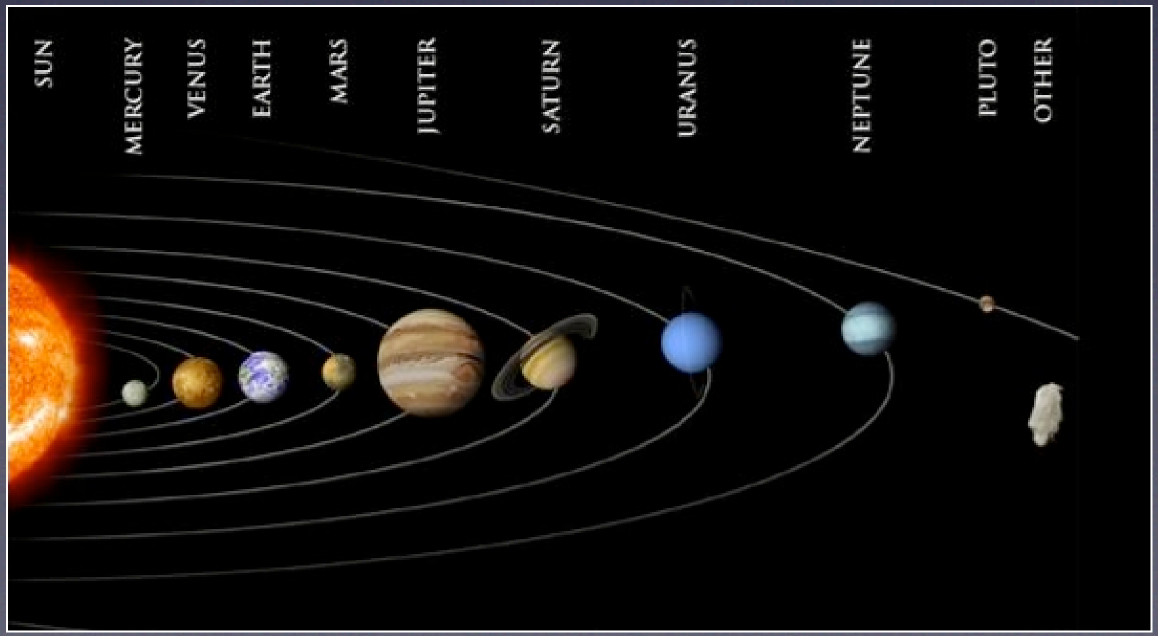


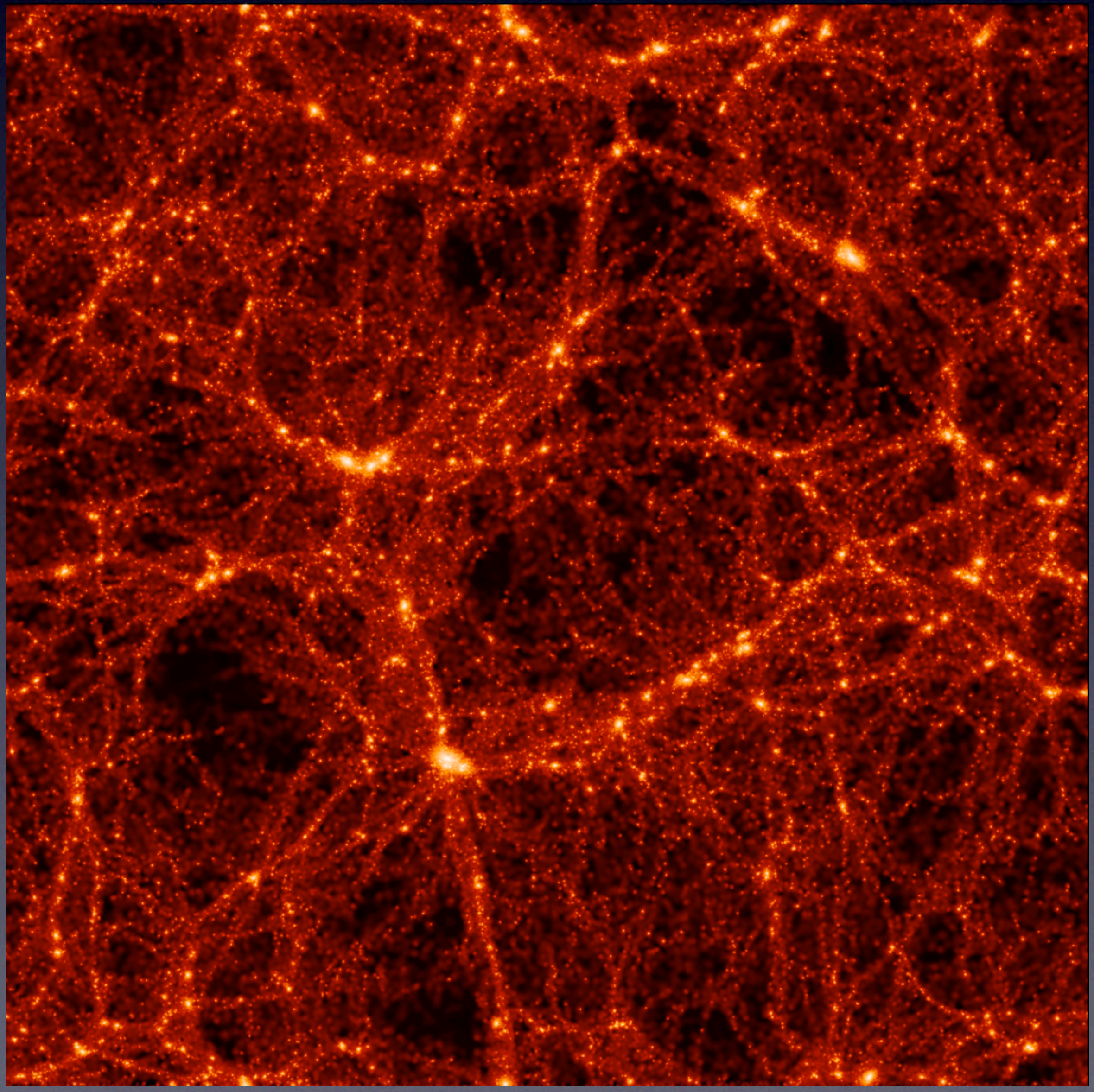
Types of N-body systems

- Stellar dynamical systems: star forming regions, star clusters, galactic nuclei
  - collisionless
  - collisional
- Fluids: Lagrangian discretization (gridless methods)

Other Types of N-body systems

- The Universe: galaxy clustering, large-scale structure formation
- Galaxies
- Proto-planetary disks
- Planetary systems





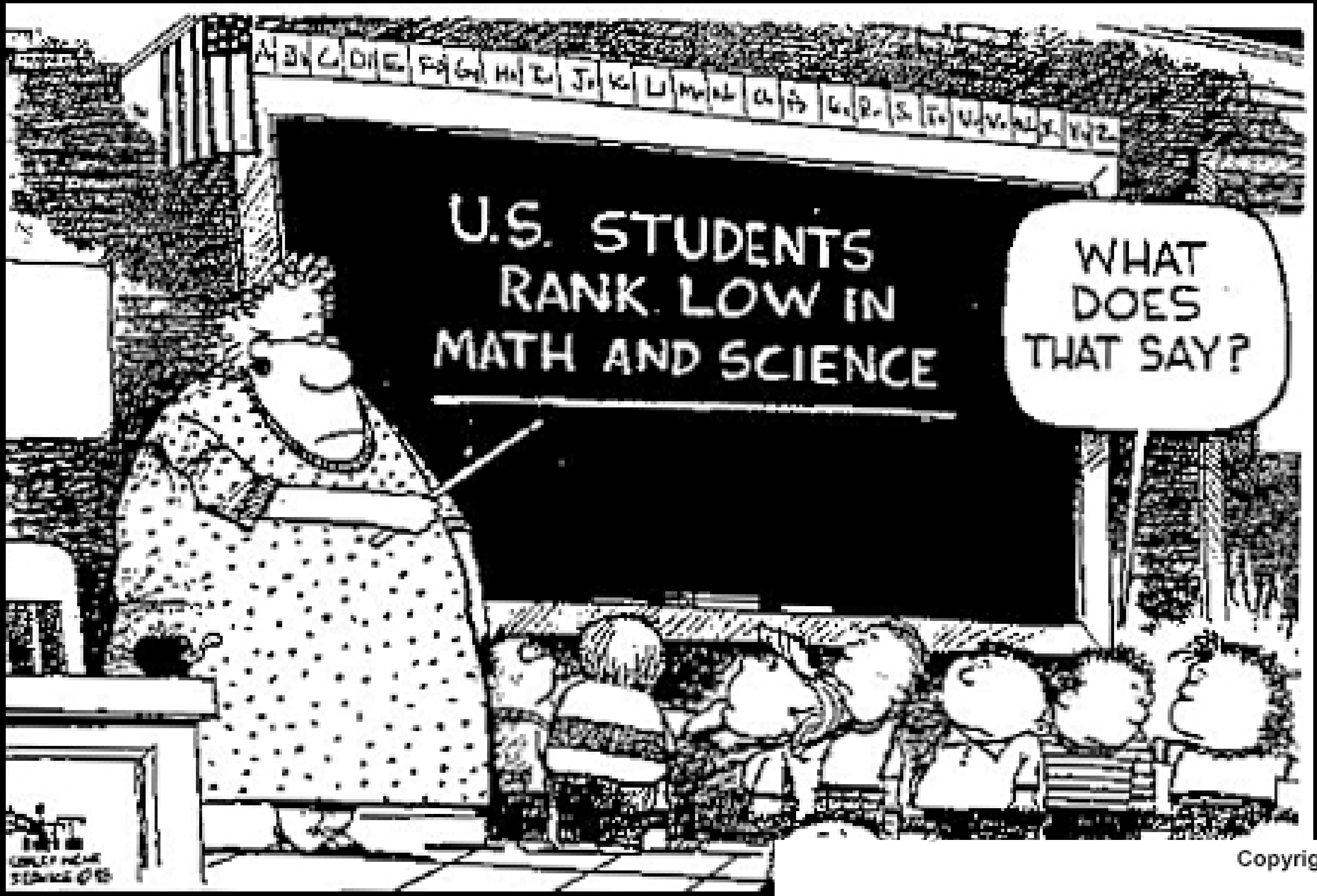
Gravity

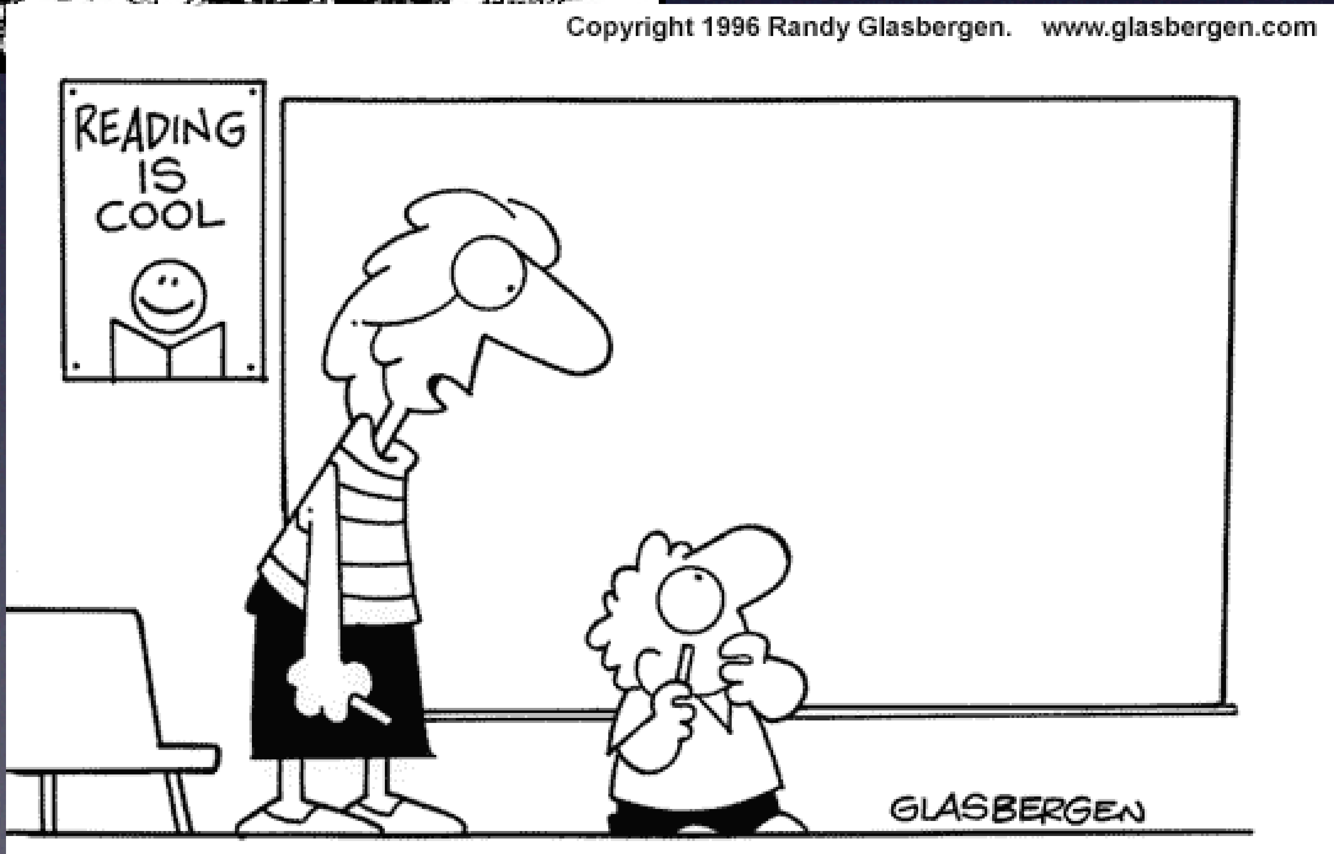
- The problem: $$\frac{d\mathbf{v}_i}{dt} = \sum_{j=1, j\neq i}^{N} \frac{G m_j \hat{\mathbf{r}}_{ij}}{r_{ij}^2}$$ $\mathcal{O}(N^2)$

- Various approaches:
  - Direct summation with smart algorithms and fast supercomputers or special-purpose computers (Japanese GRAPE)
  - Approximate methods: mean-field, Fokker-Planck, multipole, ...

Smoothed Particle Hydrodynamics

- Invented by astrophysicists in 1977 as a way of computing self-gravitating boundary-free fluid flows in 3D
- Completely gridless, particle-based scheme
- Now widely used throughout physics and engineering ("mesh-free methods")





"There aren't any icons to click. It's a chalk board."

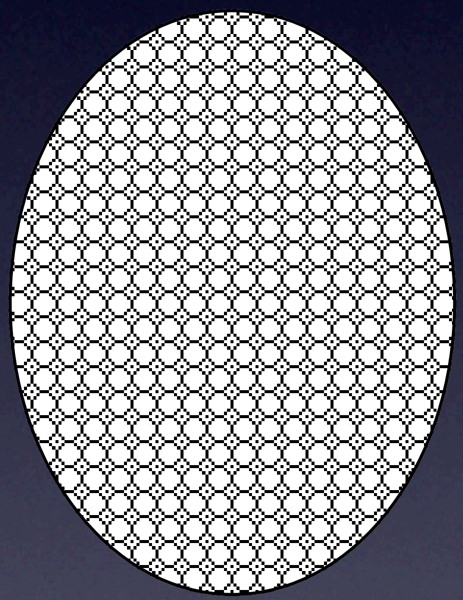
The Euler equations of motion for a perfect adiabatic fluid

$$\frac{d\mathbf{v}}{dt} = \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} = -\frac{1}{\rho}\nabla p \qquad\qquad p = A\rho^\gamma$$

can be derived from a **Lagrangian principle** with

$$L = \int \left\{ \frac{1}{2}\dot{\mathbf{x}}^2 - u[\rho(\mathbf{x})] \right\} \rho \, d^3x$$

where $u = p/[(\gamma-1)\rho]$ is the specific internal energy.



The discrete representation

$$L_{SPH} = \sum_{i=1}^{N} m_i \left[\frac{1}{2}\dot{\mathbf{x}}_i^2 - u(\rho_i) \right]$$

with

$$\rho_i = \sum_j m_j W_{ij} \qquad\qquad W_{ij} = W(|\mathbf{x}_i - \mathbf{x}_j|;\, h)$$

is used in the Smoothed Particle Hydrodynamics (SPH) techniques.

SPH **equations of motion** for adiabatic flow:

$$\boxed{\frac{d\mathbf{v}_i}{dt} = -\sum_j m_j \left(\frac{p_i}{\rho_i^2} + \frac{p_j}{\rho_j^2} \right) \nabla_i W_{ij}}$$

$$\frac{d\mathbf{x}_i}{dt} = \mathbf{v}_i$$

where

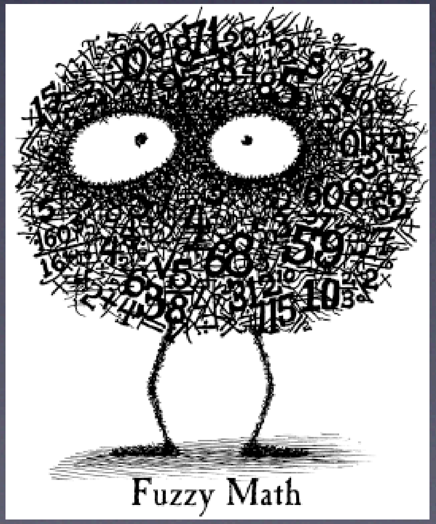
$$\rho_i = \sum_j m_j W_{ij} \qquad\qquad p_i = A_i \rho_i^\gamma$$

The following **energy and momentum conservation** laws are satisfied **exactly** by the above equations:

$$\frac{d}{dt}\left(\sum_{i=1}^{N} m_i \mathbf{v}_i \right) = 0$$

$$\frac{d}{dt}\left(\sum_{i=1}^{N} m_i \left[\frac{1}{2} v_i^2 + u_i \right] \right) = 0$$

where $u_i = p_i / [(\gamma - 1)\rho_i]$.



Example 1: hydrodynamic calculations of stellar collisions

- Why? What? Where?
- Smoothed Particle Hydrodynamics (SPH)
- Direct summation gravity (GRAPE)
- With N. Ivanova, J. Lombardi, REU students

Stars collide!

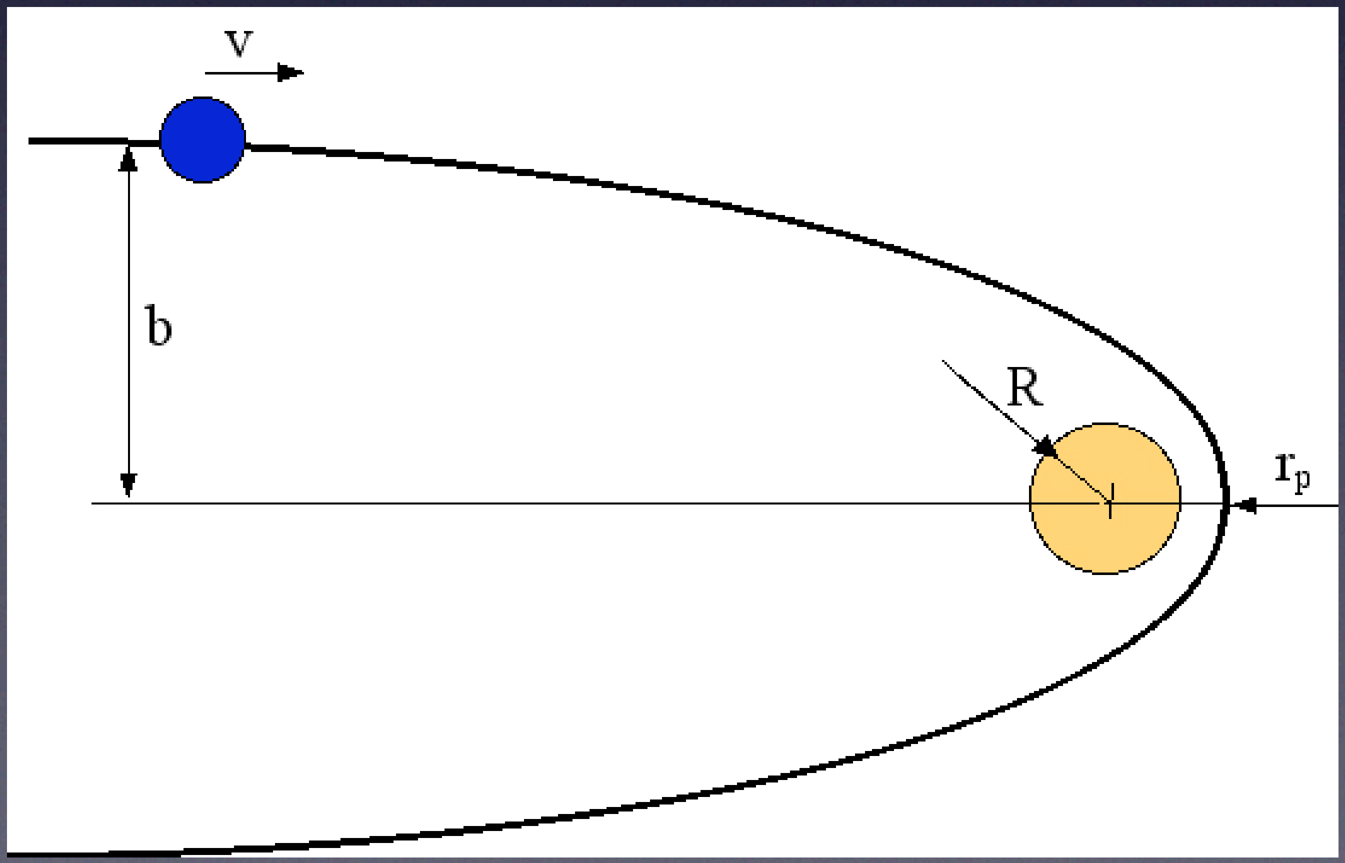
$$\tau_c \sim \boxed{10^{10}\,\mathrm{yr}} \left(\frac{\rho}{10^6\, M_\odot\, \mathrm{pc}^{-3}} \right)^{-1} \left(\frac{v}{10\,\mathrm{km\,s^{-1}}} \right) \left(\frac{R}{R_\odot} \right)^{-1}$$

$$\theta = \frac{2Gm}{Rv^2} \gg 1 \quad \text{Safronov Number}$$

$$b^2 = r_p^2 \left(1 + \frac{2Gm}{r_p v^2} \right)$$

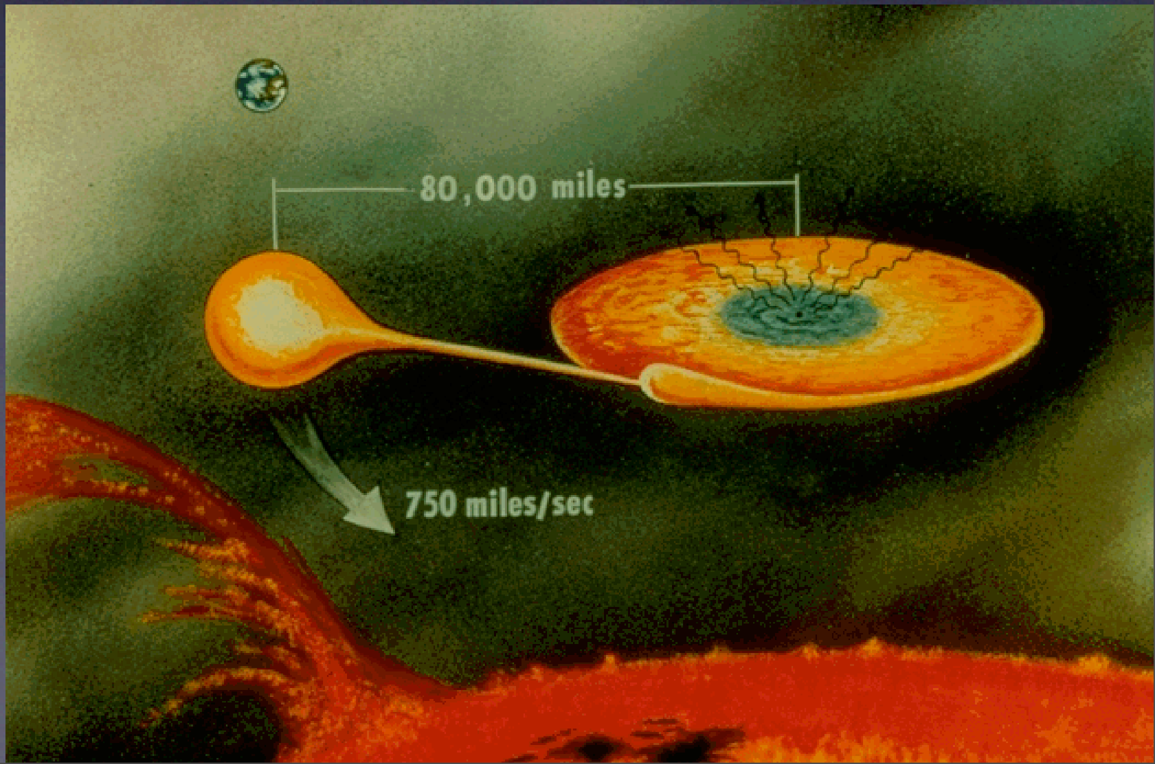
$$\sigma_c = \pi b^2 (r_p = R)$$

$$\tau_c \sim (n \sigma_c v)^{-1}$$



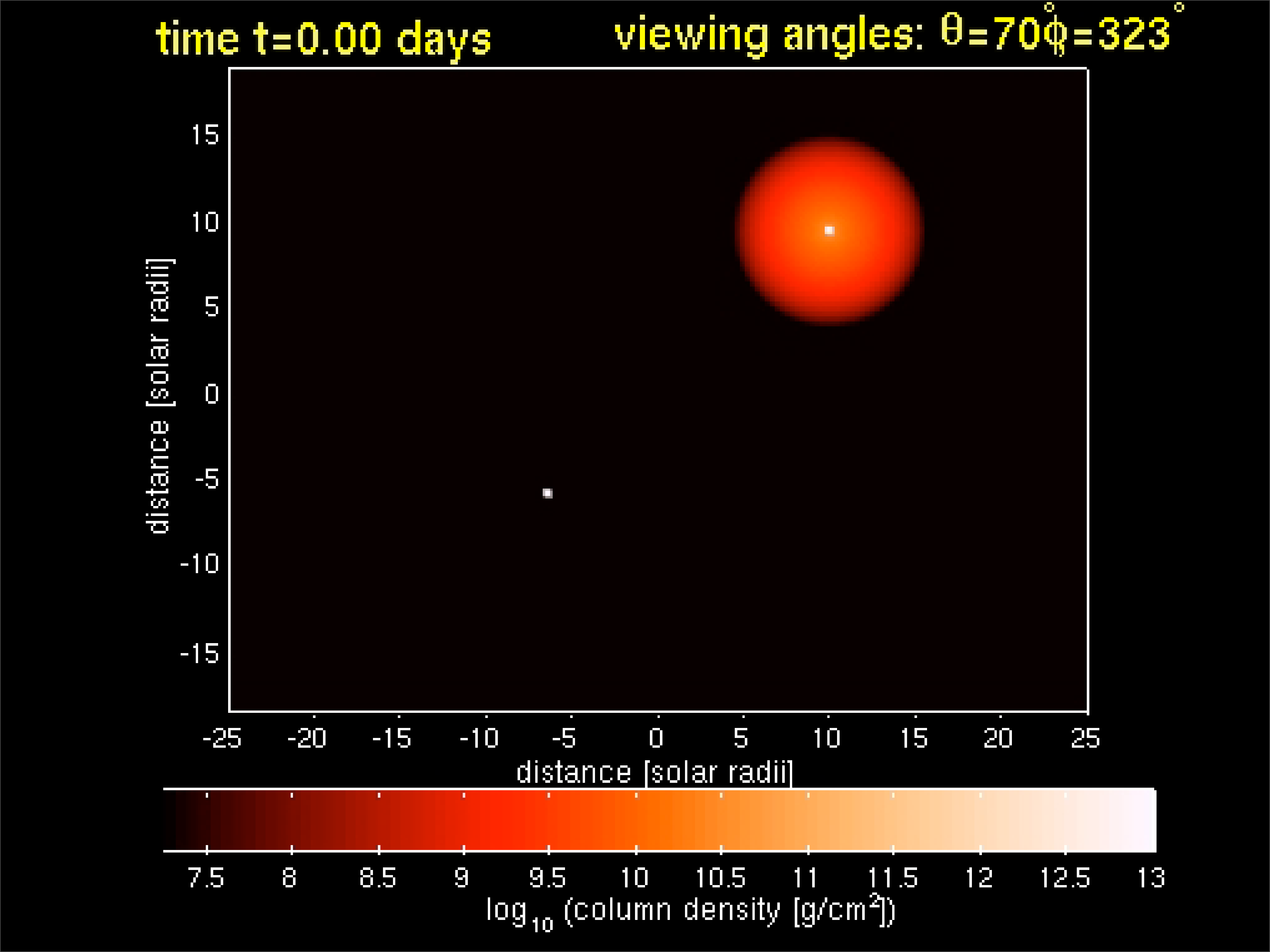
Ultracompact X-ray Binaries

- Very bright X-ray sources ($L_{\mathrm{x}} \sim 10^4\, L_{\odot}$)
- Neutron stars accreting from companion stars with P_{orb} as short as minutes !
- Only plausible companion type (mass donor) is a white dwarf
- Produced at enormous rates in dense star clusters



Neutron star collisions

- White dwarf is core of red giant
- Red giants and neutron stars are massive and concentrate near the high-density center of a star cluster
- Red giants have large envelopes that collide frequently and are easily disrupted



time t=0.00 days
viewing angles: θ=70° φ=323°
distance [solar radii]
distance [solar radii]
log10 (column density [g/cm2])

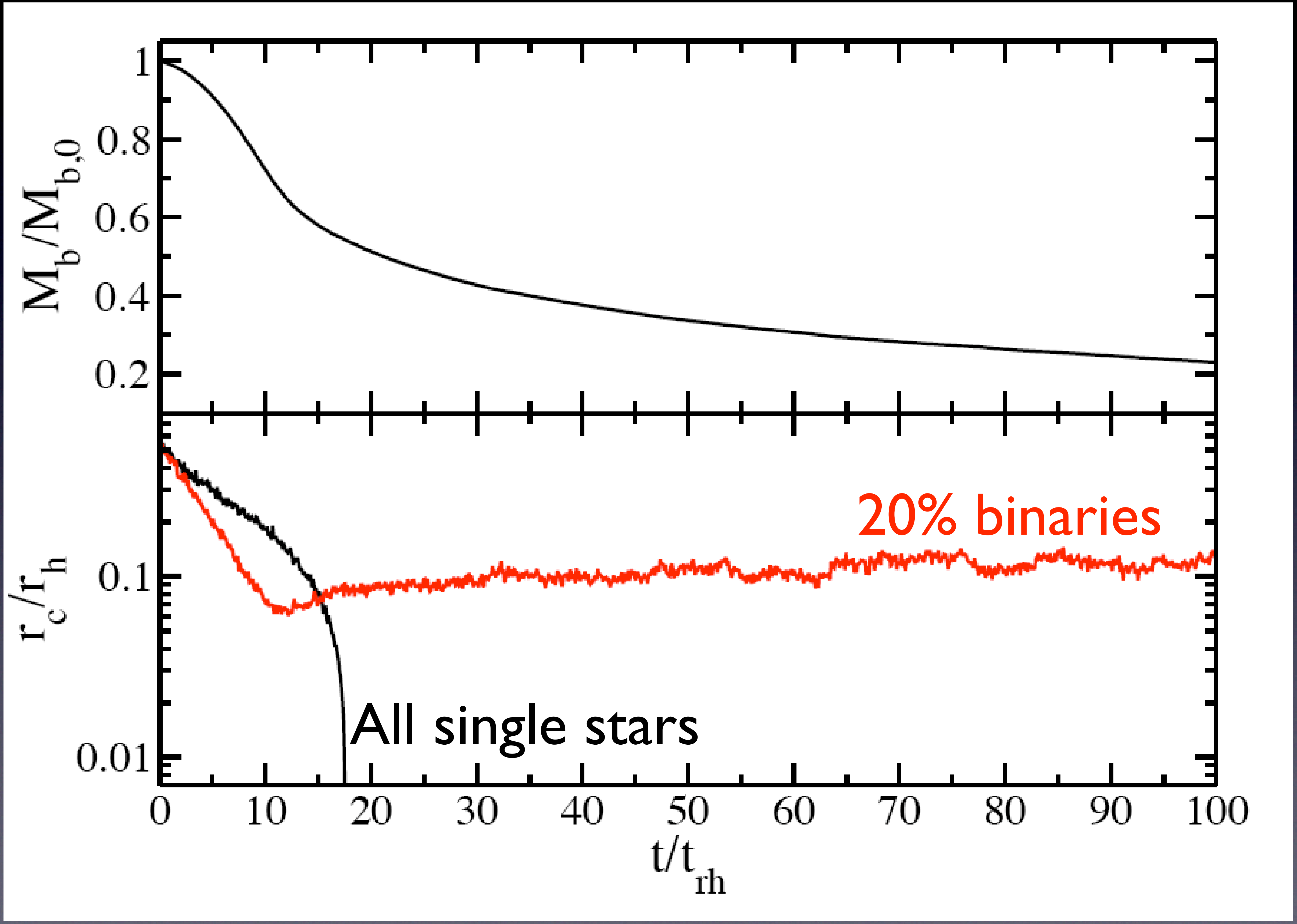
Example II: avoiding the gravothermal catastrophe in star clusters

- Why? What? Where?
- Fokker-Planck approximation
- Monte Carlo method in spherical symmetry
- With J. Fregeau, A. Gürkan

Stars vs Star Clusters

- Spherical, self-gravitating (fluid vs collisionless)
- Energy transport from hot interior to surface (radiation/convection vs diffusion)
- Energy loss timescale << ages
- $\longrightarrow$ need energy source (nuclear burning vs WHAT?)

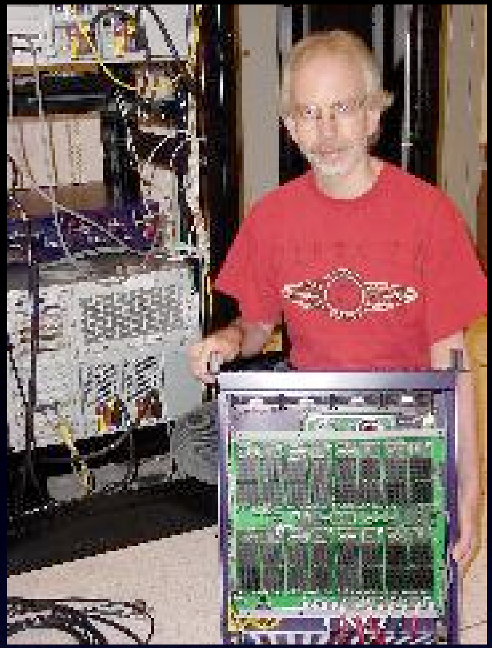
Binary burning



Summary

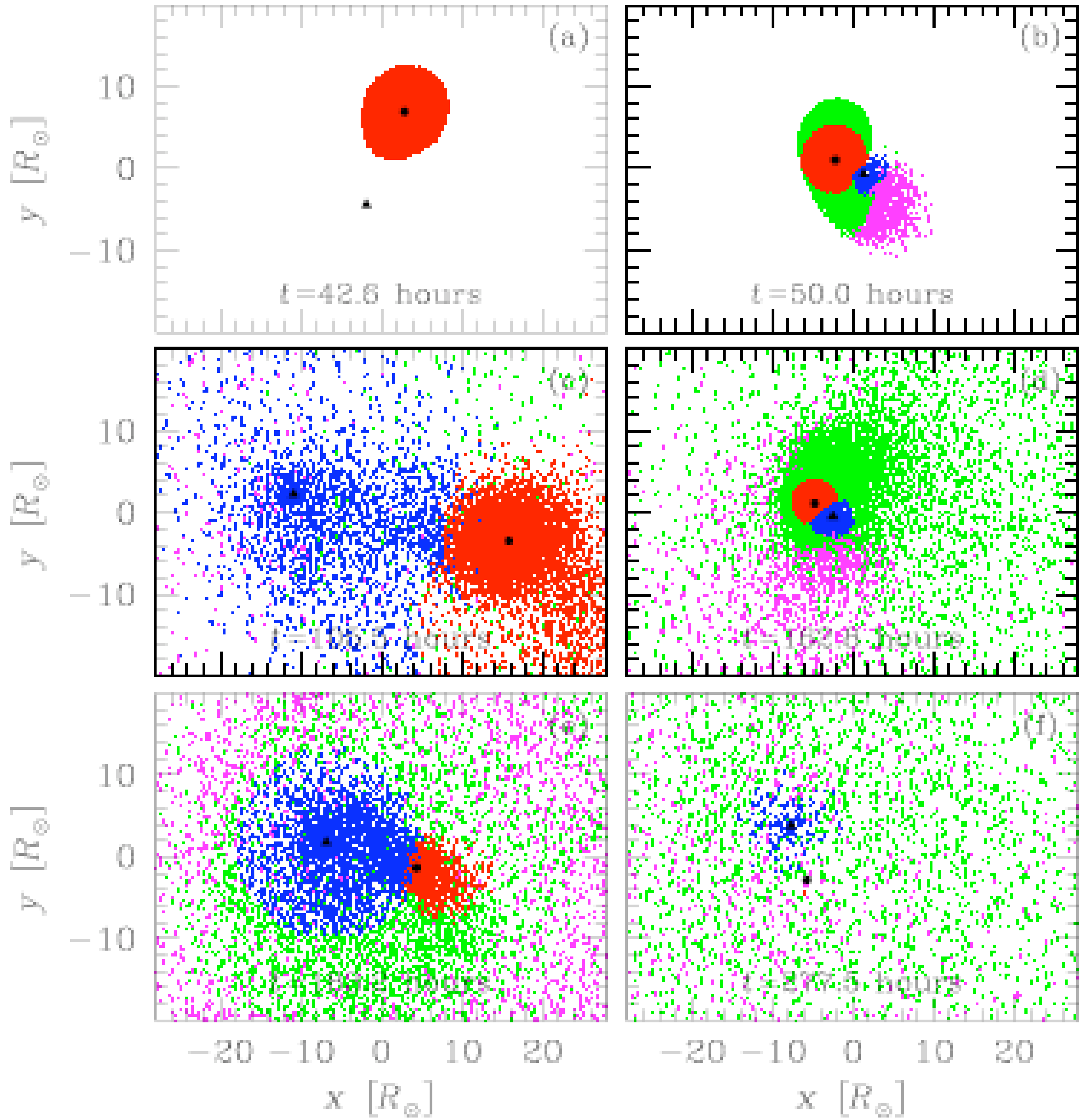
- Particle simulations play a central role in theoretical astrophysics, key to understanding physical processes across wide range of scales
- Gravity and complexity of interactions (physical collisions, 3-body, 4-body) make astrophysical N-body simulations very challenging

Brute force N-body ...



- Special-purpose supercomputing (parallel clusters and/or accelerators such as GRAPE boards

- Smart algorithms:
  - Individual timesteps
  - Regularization
  - Tree/multipole expansions
  - Parallelization


(a)
t = 42.6 hours
(b)
t = 50.0 hours
(f)
10
0
−10
y [R⊙]
−20 −10 0 10 20
x [R⊙]

Fokker-Planck Limit

- Weakly collisional limit $$\tau_{\rm rel} \sim \frac{N}{\ln N}\,\tau_{\rm dyn} \gg \tau_{\rm dyn} \qquad (\theta \gg 1)$$
- Mean field calculation $$\nabla^2 \phi = 4\pi G \langle \rho \rangle$$
- Dynamical equilibrium $$\phi(\mathbf{x}, t) = \phi(\mathbf{x})$$

Monte Carlo

- In spherical symmetry each orbit is fully specified by E, J in potential $\phi(r)$
- Each particle i=1,...,N in system has

$$E_i,\ J_i,\ m_i, \cdots$$

- Interactions computed in Fokker-Planck approximation by Monte Carlo realization of small-angle scatterings

Binary burning

- Most stars are in bound pairs (binaries)
- In star clusters “hard” binaries are long lived and act as a source of kinetic energy (“heat”)
- Heggie’s theorem: “hard binaries get harder”
- Computational challenge: interactions of binaries (3-body, 4-body) are often resonant and extremely hard to compute
- Current research: use Monte Carlo method with full 3-body and 4-body integrations to calculate long-term evolution of star clusters

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A numerical approach to the testing of the fission hypothesis

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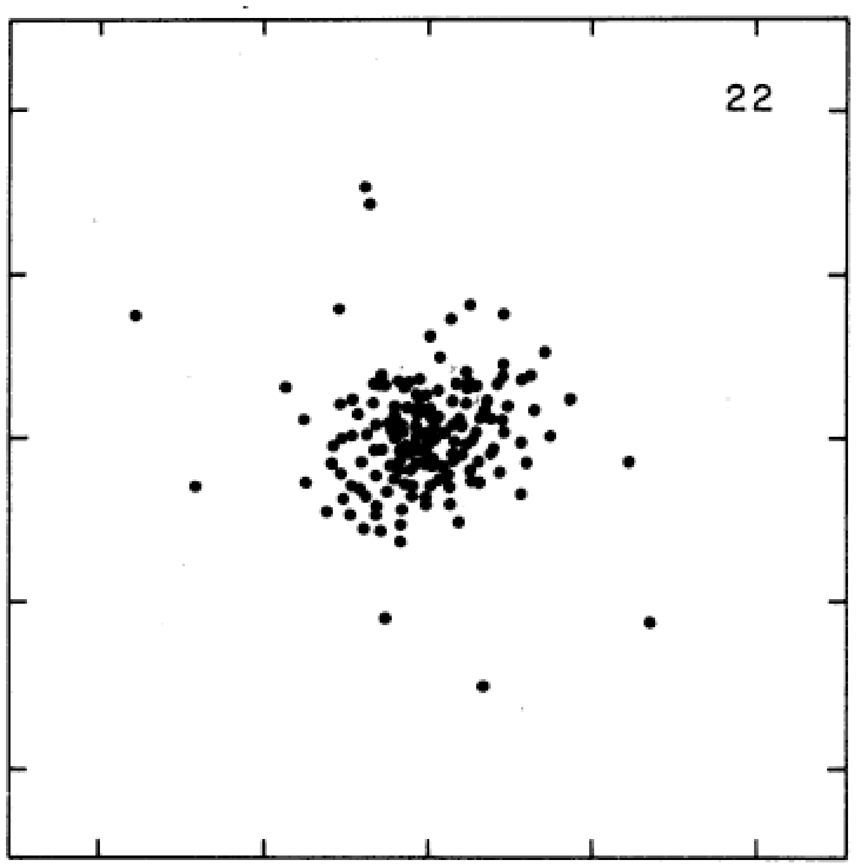
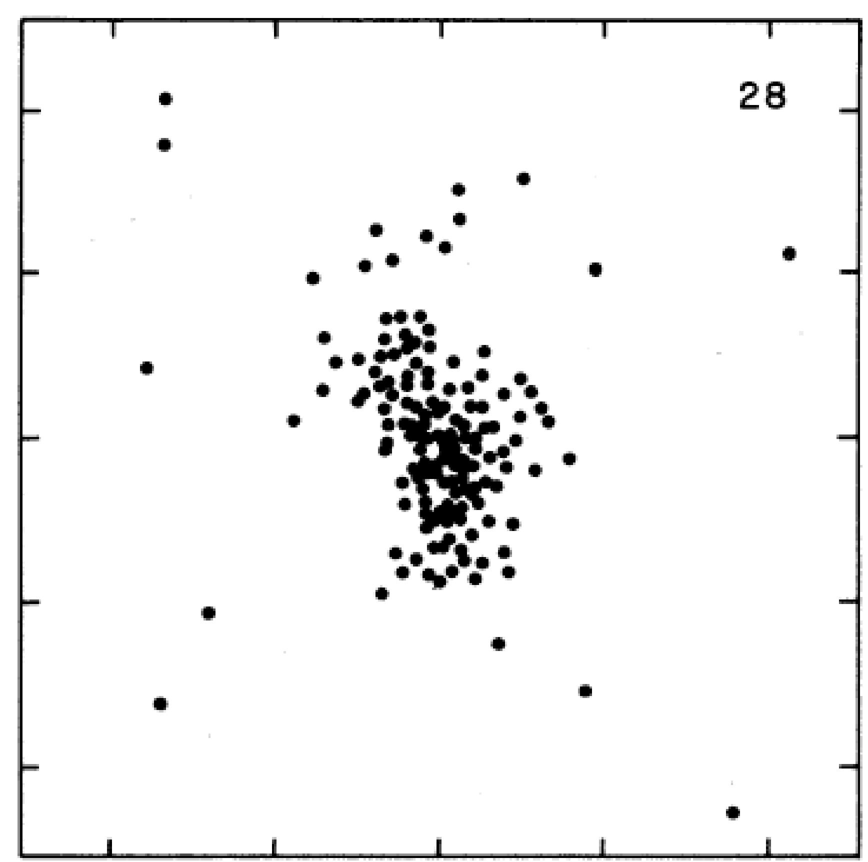
(Received 12 August 1977; revised 16 September 1977)

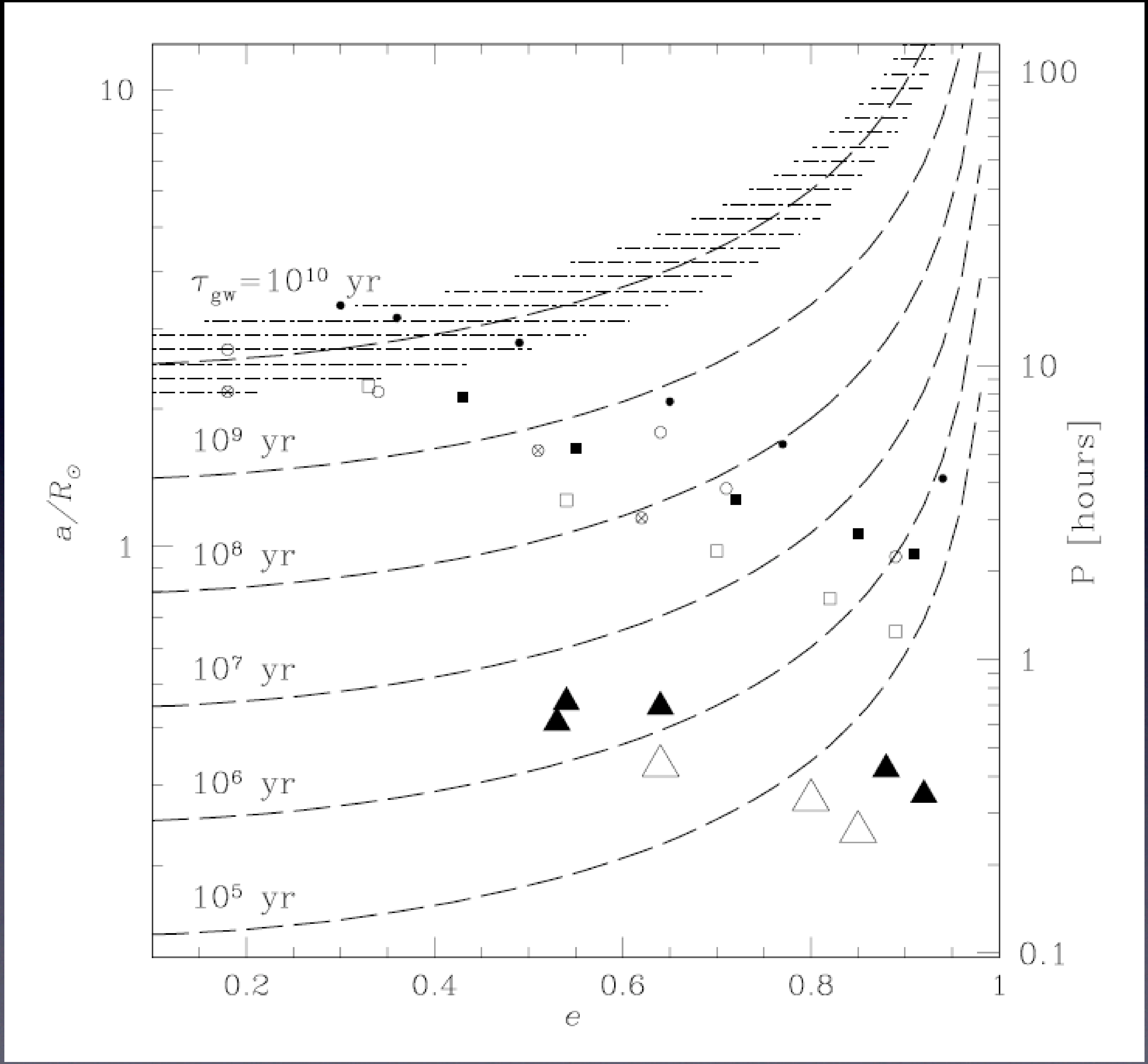
A finite-size particle scheme for the numerical solution of two- and three-dimensional gas dynamical problems of astronomical interest is described and tested. The scheme is then applied to the fission problem for optically thick protostars. Results are given, showing the evolution of one such protostar from an initial state as a single, rotating star to a final state as a triple system whose components contain 60% of the original mass. The decisiveness of this numerical test of the fission hypothesis and its relevance to observed binaries are briefly discussed.



(a)



(b)



$\tau_{gw}=10^{10}$ yr
10^9 yr
10^8 yr
10^7 yr
10^6 yr
10^5 yr
10
1
0.2
0.4
0.6
0.8
1
e
$a/R_\odot$
100
10
1
0.1
P [hours]

The Gravothermal Catastrophe