Index Theorem and Random Matrix Theory for Improved Staggered Quarks

E. Follana

University of Glasgow

In collaboration with:

C. Davies (University of Glasgow)

A. Hart (University of Edinburgh)

HPQCD and UKQCD

- (hep-lat/0406010, E. F., A. Hart and C.T.H. Davies.)
- U. Wenger, “Comparative study of overlap and staggered fermions in QCD”.
  (hep-lat/0406027, S. Dürr, C. Hoelbling and U. Wenger.)
- Kit Yan Wong, “Topology and staggered fermion action improvement”.

Outline

- Motivation
- Staggered Dirac operators
- Index Theorem
- Low-lying spectrum and RMT predictions

Motivation

- Topological features of QCD:
  - Axial anomaly
  - $\eta^{'}$ mass
  - Predictions of RMT for each sector of Q in the ϵ regime
- The low lying spectrum of staggered fermions appeared to be "topology blind".
- But near the continuum we should see the correct behaviour.
- Improved staggered actions are being used today in large-scale dynamical simulations.
- It is therefore important to understand to what extent staggered quarks show the correct topological properties.
- Use the improved staggered formulations to address these questions:
  - Can we see the continuum features in lattices with reasonable parameters?
  - Are (improved) staggered quarks sensitive to topology?
  - Can we reproduce the detailed predictions of RMT?

Staggered Dirac Operators

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One-link (naive, KS) staggered Dirac operator

$$\begin{aligned} \mathrm{D}(x,y) &= \frac{1}{2u_0} \sum_{\mu} \alpha_\mu(x) \left(U_\mu(x) \delta_{x+\mu,y} - U_\mu^\dagger(y) \delta_{x,y+\mu} \right) \\ &= \mathrm{D}_{e,o}(x,y) + \mathrm{D}_{o,e}(x,y) \end{aligned}$$

$$\alpha_\mu(x) = (-1)^{\sum_{\nu \mu} x_\nu}$$

Staggered Dirac Operators

- Antihermitian

$$\mathrm{D}^\dagger = -\,\mathrm{D} \Longrightarrow -\,\mathrm{D}^\dagger\,\mathrm{D} \geq 0 \text{ , hermitian}$$

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  Unphysical taste-changing interactions, involving at leading order the exchange of a gluon of momentum $q \approx \pi/a$.

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  Unphysical taste-changing interactions, involving at leading order the exchange of a gluon of momentum $q \approx \pi/a$.

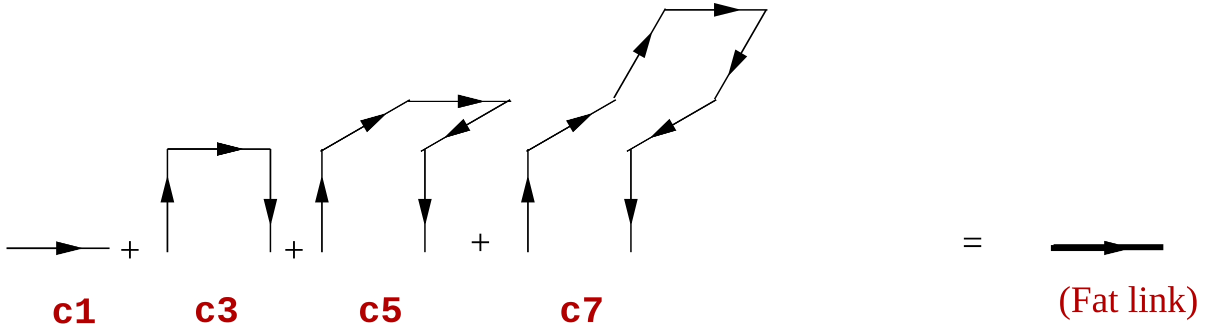
- Such interactions are perturbative for typical values of the lattice spacing, and can be corrected systematically a la Symanzik.

Improved Staggered Actions

Smear the gauge field to remove coupling between quarks and gluons with momentum π/a.

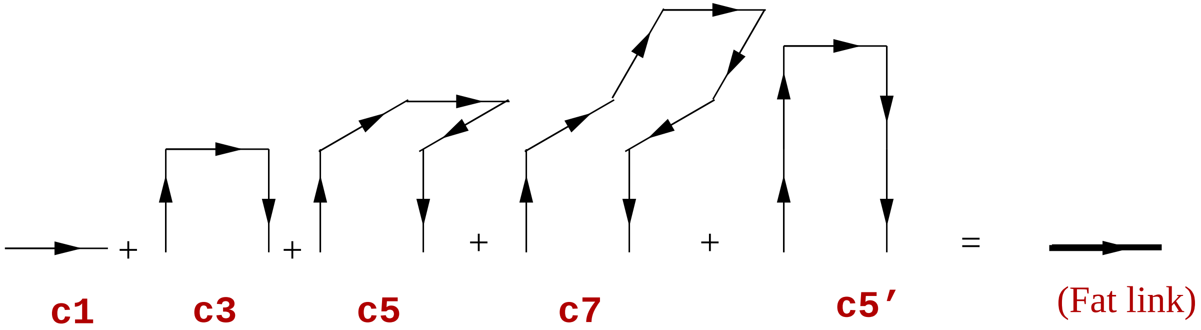
Improved Staggered Actions

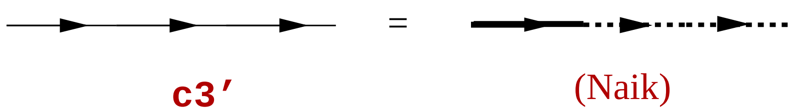
- FAT7(TAD)



Improved Staggered Actions

- ASQ(TAD)



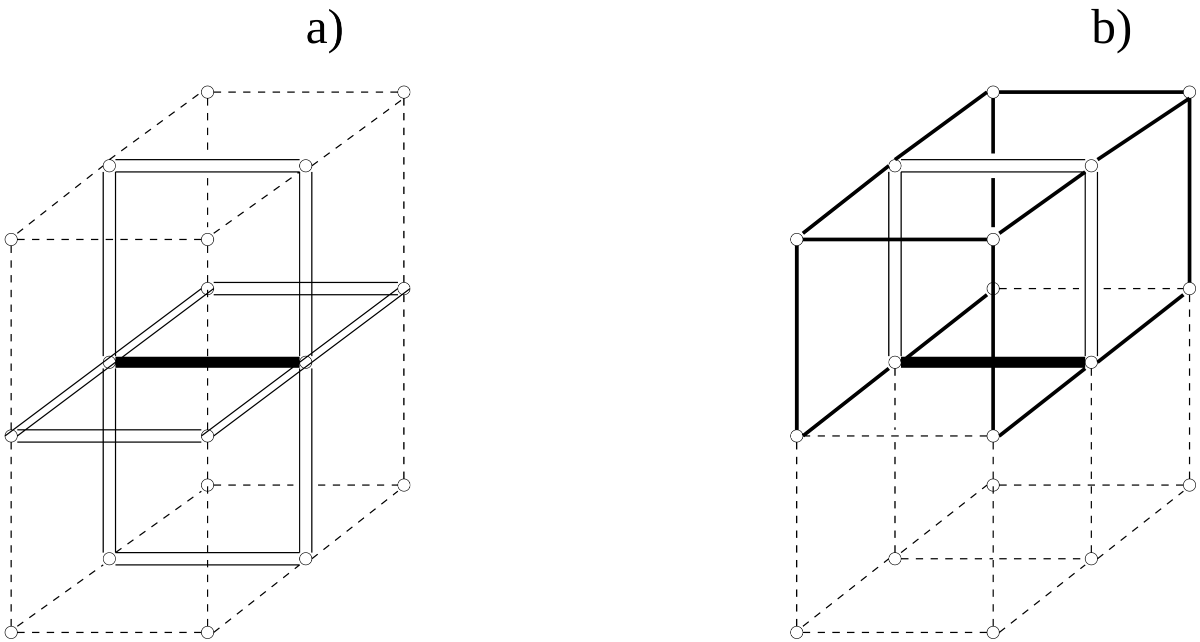


(S. Naik, the MILC collaboration, P. Lepage.)

Improved Staggered Actions

- HYP (Hypercubic Blocking)

Three levels of (restricted) APE smearing with projection onto $SU(3)$ at each level.
Each fat link includes contributions only from thin links belonging to hypercubes attached to the original link.



(A. Hasenfratz, F. Knechtli.)

Improved Staggered Actions

- HISQ (Highly Improved Staggered Quarks)

Two levels of smearing: first a FAT7 smearing on the original links, followed by a projection onto $SU(3)$, then ASQ on these links.

$$\text{FAT7}\|_{SU(3)} \otimes \text{ASQ}$$

(E.F., Q. Mason, C. Davies, K. Hornbostel, P. Lepage, H. Trottier.)

Improved Staggered Actions

- ASQTAD and HISQ improvement removes a^2 errors at tree level.
- HYP and HISQ show very small taste-changing effects.

Index Theorem (Continuum)

- For QCD in the continuum, the topological charge is given by

$$Q = \frac{1}{32\pi^2} \int d^4x \; \epsilon_{\mu\nu\sigma\tau} \; \mathrm{tr} F_{\mu\nu}(x) F_{\sigma\tau}(x)$$

- Atiyah-Singer Index theorem:

$$Q = \frac{m}{n_f} \; \mathrm{tr}(\gamma_5 S_F) = \frac{m^2}{n_f} \sum_n \frac{\langle n|\gamma_5|n\rangle}{\lambda^2 + m^2} = n^+ - n^-$$

where $|n\rangle$ are the eigenfunctions of the Dirac operator in the given gauge field background, and n^+, n^- are the number of positive and negative chirality zero modes

$$\gamma_5 |n\rangle = \pm |n\rangle$$

Index Theorem (Lattice)

- Dirac operators which satisfy the Ginsparg-Wilson relation

$$\{\gamma_5, \mathrm{D}\} = \overline{a}\,\mathrm{D}\,\gamma_5\,\mathrm{D}$$

can have exact, chiral zero-modes, which can then be used to define a topological charge via the identity

$$Q = a^4 \sum_x q(x) = n^+ - n^-$$

$$q(x) = -\frac{1}{2}\,\overline{a}\,\mathrm{tr}\,\{\gamma_5\,\mathrm{D}(x,x)\}$$

where $q(x)$ is a local, gauge invariant function of the gauge fields.

- For the fixed point Dirac operator, furthermore,

$$Q^{FP} = n^+ - n^-$$

where Q^{FP} is the fixed point topological charge.

(P. Hasenfratz, V. Laliena, F. Niedermayer.)

Index Theorem (Lattice)

- The staggered Dirac operator has no exact zero modes, therefore we cannot expect an exact index theorem.
  But close to the continuum limit, we expect to see a similar behaviour: the first few eigenmodes of high chirality, in the number required by the continuum index theorem, and the rest of the eigenmodes with small chirality.
- It has been seen that if you smooth the configurations enough, for example by repeatedly smearing, eventually the continuum features appear. (P. Damgaard, U. Heller, R. Niclasen, K. Rummukainen.)
- That is also the case, if we study lattice discretizations of continuum instantons. (J. Smit, J. Vink.)
- But, we want to study the features of the raw, non-smoothed configurations.
- Chirality must be measured using a taste-singlet operator, which is the one that couples to the anomaly.

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$$\varkappa_n = \langle n|\gamma_5|n\rangle$$

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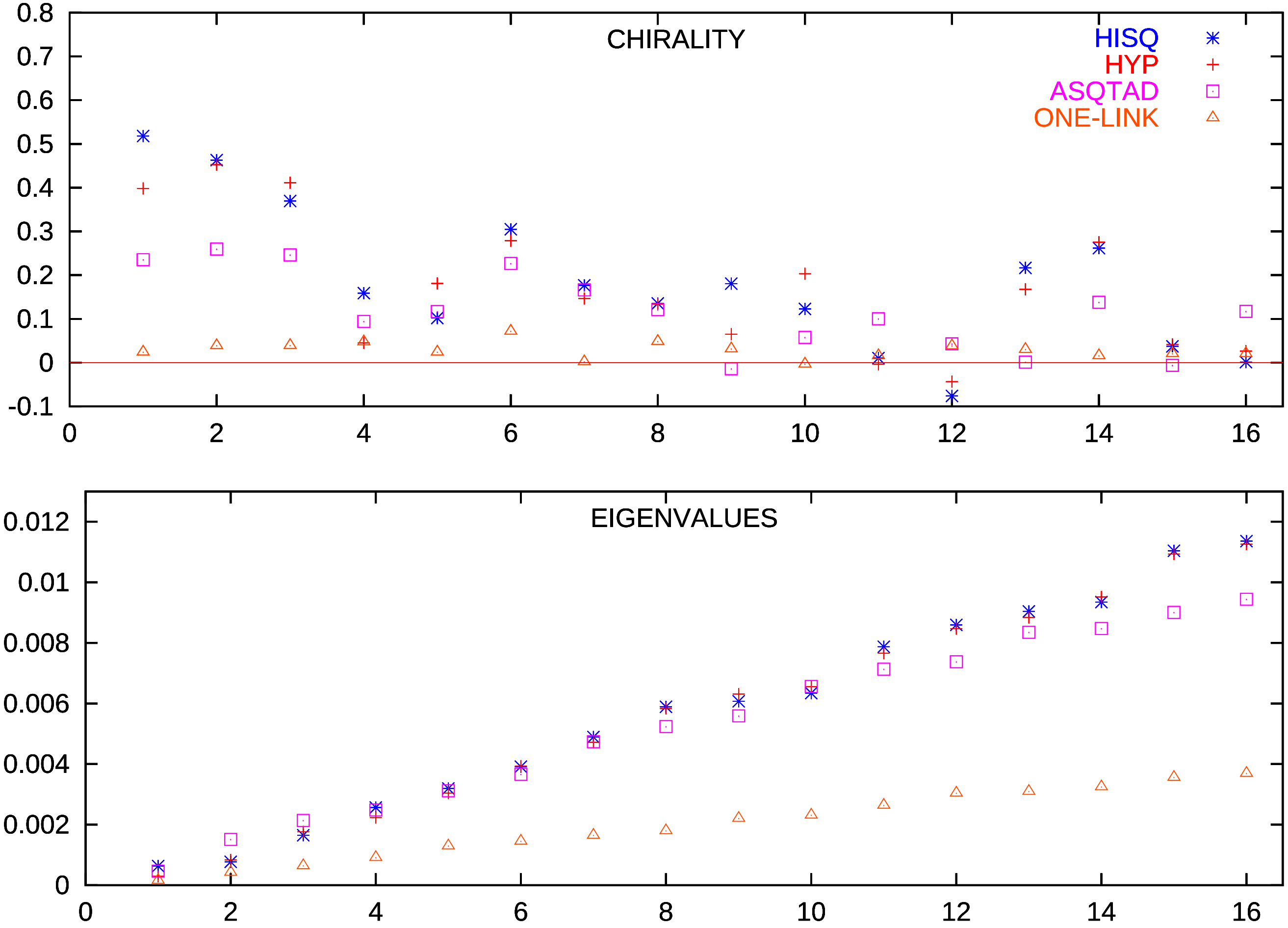
- Are there:
  - $n_t \; n^+$ near-zero modes with chirality $\varkappa \approx 1$
  - and/or $n_t \; n^-$ near-zero modes with chirality $\varkappa \approx -1$

such that $$Q_{gl} \approx \frac{1}{n_t} \left(n^+ - n^- \right) \quad ?$$

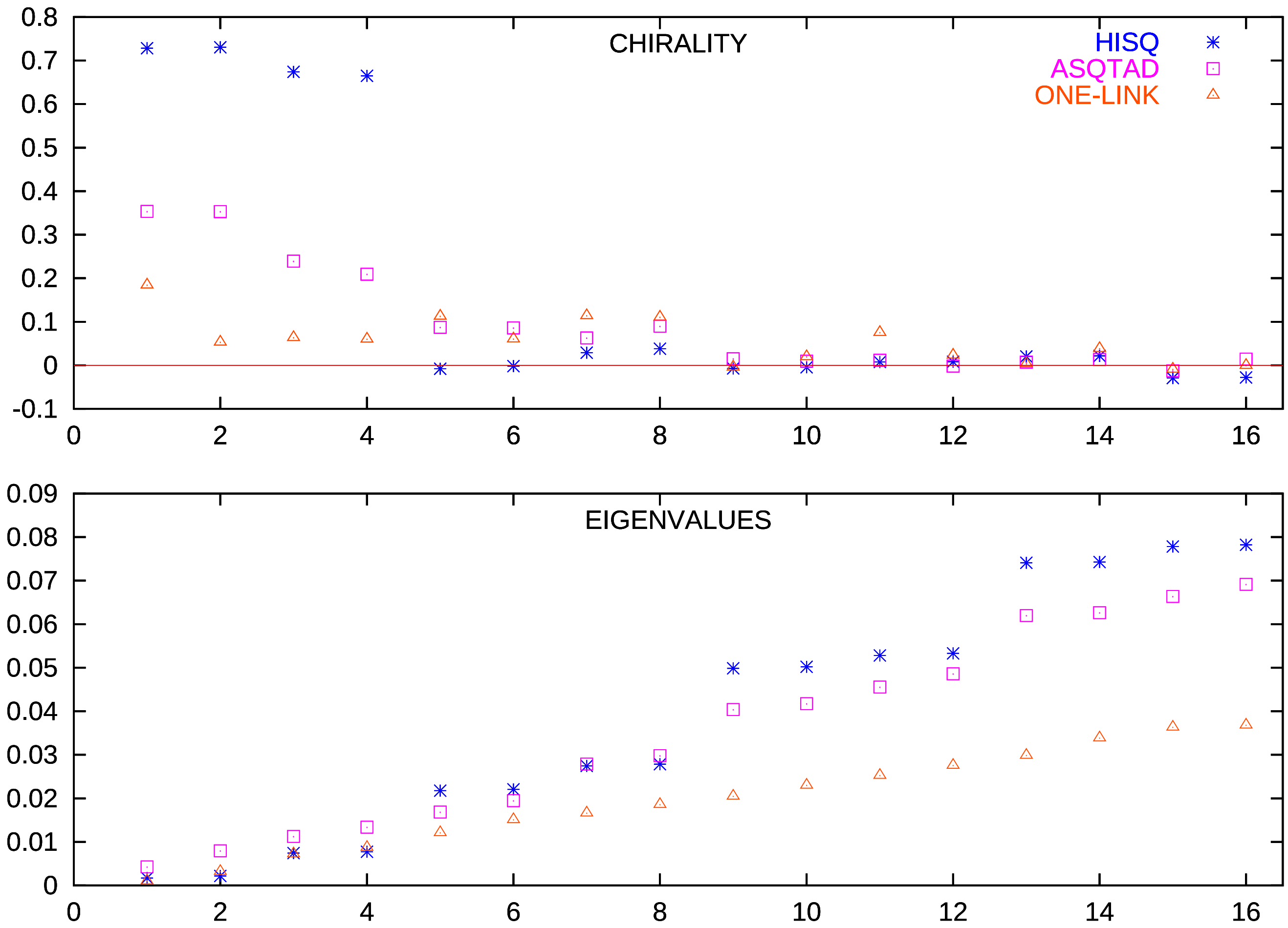
(Almost) Index Theorem for Improved Staggered Quarks

- Unimproved glue, $Q_{gl} = 2$, $a \approx .1$ fm , $16^3\,32$



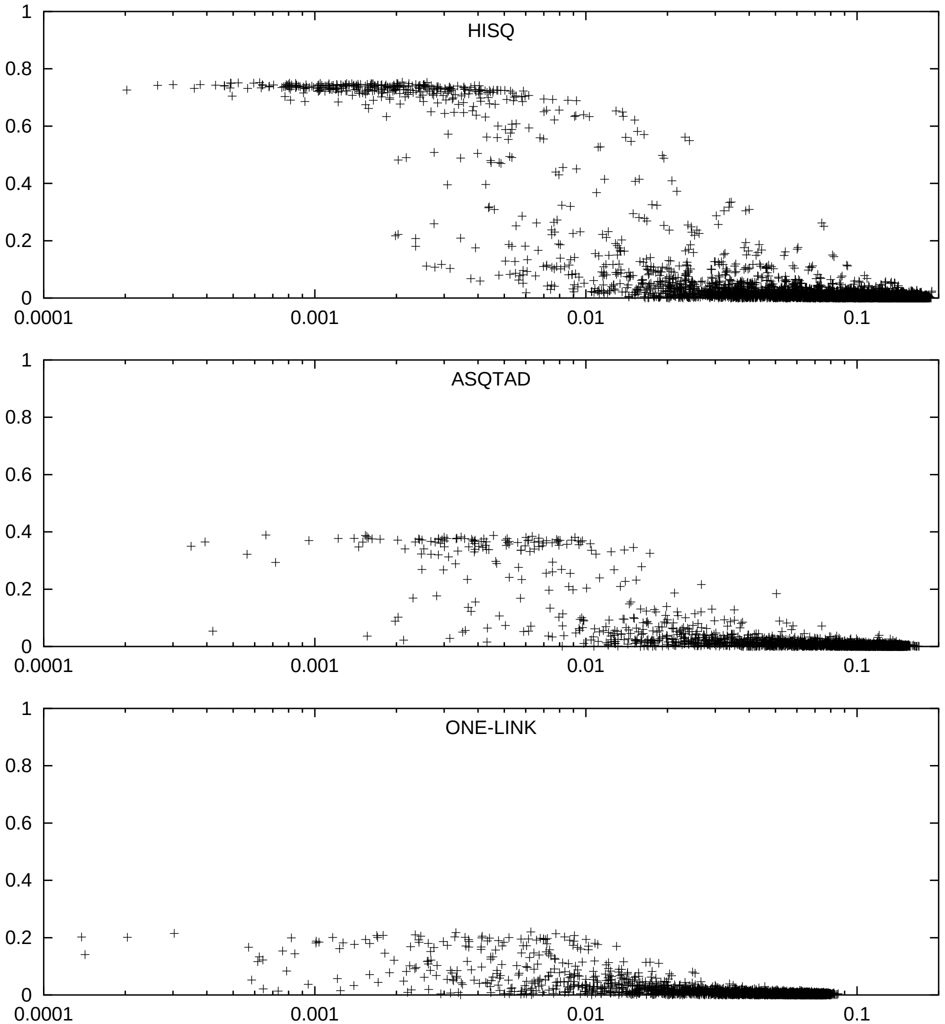
(Almost) Index Theorem for Improved Staggered Quarks

- Improved glue, $Q_{gl} = 2$, $a \approx .09$ fm , 16^4



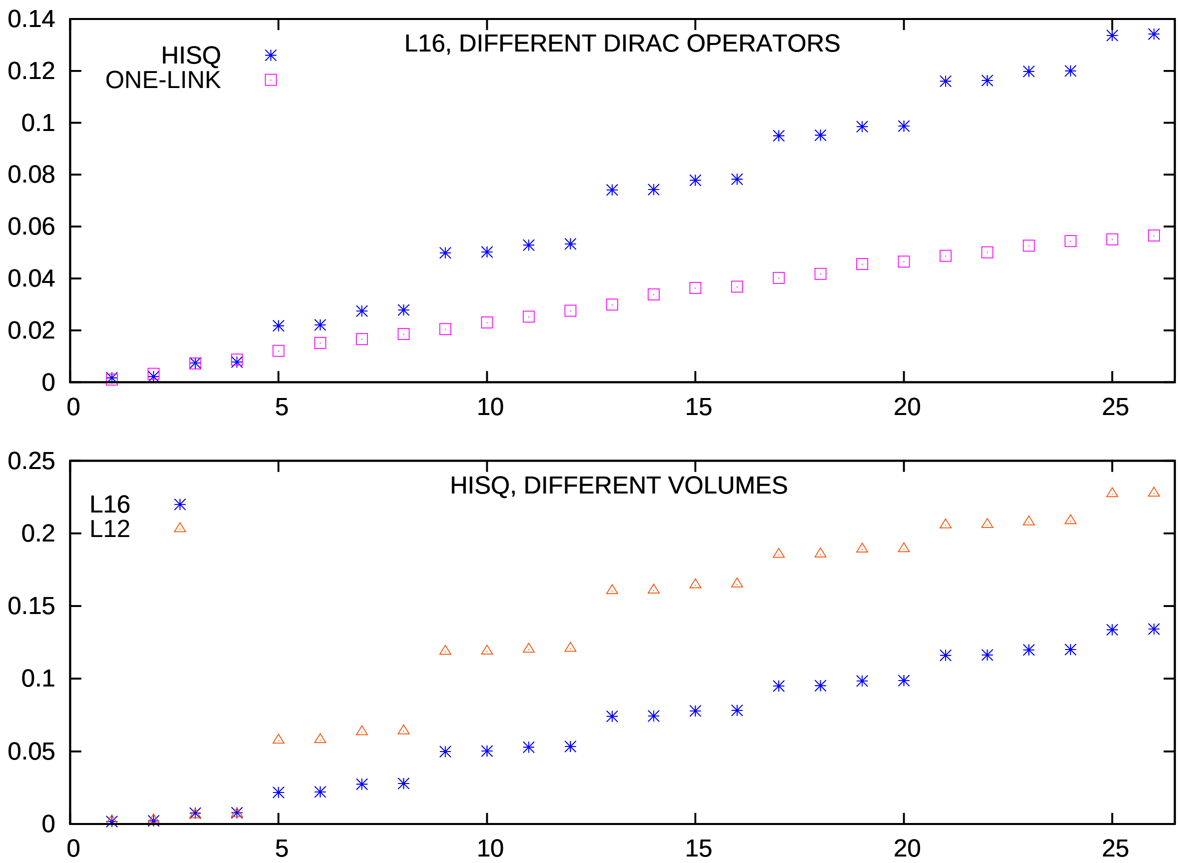
(Almost) Index Theorem for Improved Staggered Quarks

- Scatter plot: chirality vs eigenvalue.



Spectrum Degeneracy

- We expect to see a degeneracy of n_t near the continuum limit.
- Tree level improved glue, $a \approx 0.09$ fm



RMT and Staggered Quarks

- RMT predicts, for each sector of fixed topological charge Q, the probability distribution of the non-zero low lying eigenvalues of the Dirac operator up to a constant.

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- RMT predicts, for each sector of fixed topological charge Q, the probability distribution of the non-zero low lying eigenvalues of the Dirac operator up to a constant.
- In past studies, the results obtained for staggered quarks in any sector of topological charge Q were identical to the one predicted for the sector $Q = 0$, really topologically blind!

Eigenvalue Averages

- Ordering the non-zero eigenvalues $\{\pm i\lambda\}$ as

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots$$

the ratios of the average of each eigenvalue at fixed Q

$$\frac{\langle\lambda_j\rangle_Q}{\langle\lambda_k\rangle_Q}$$

can be calculated from RMT, for different values of Q, j and k, and compared with the results of numerical simulations.

Eigenvalue Averages

- We calculate Q_{glue} by cooling, using two different cooling actions and an improved gluonic topological charge operator. We throw away configurations whose charge is unstable under cooling or does not agree between the two actions (10% of the total.)

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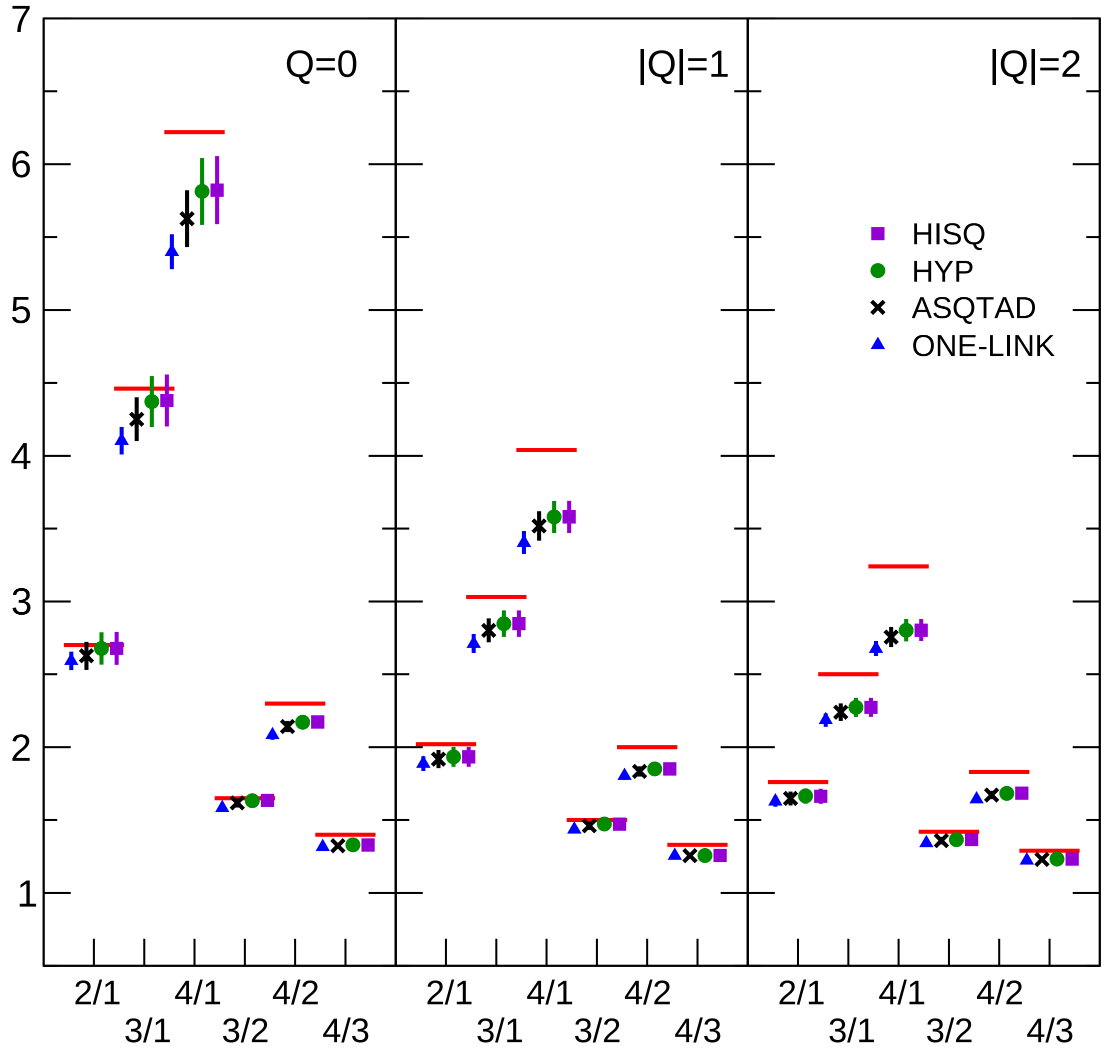
- We calculate the ensemble average of Λ_j, at fixed topological charge, and compare the corresponding ratios with the predictions of RMT.

$$\frac{\langle \Lambda_j \rangle_Q}{\langle \Lambda_k \rangle_Q}$$

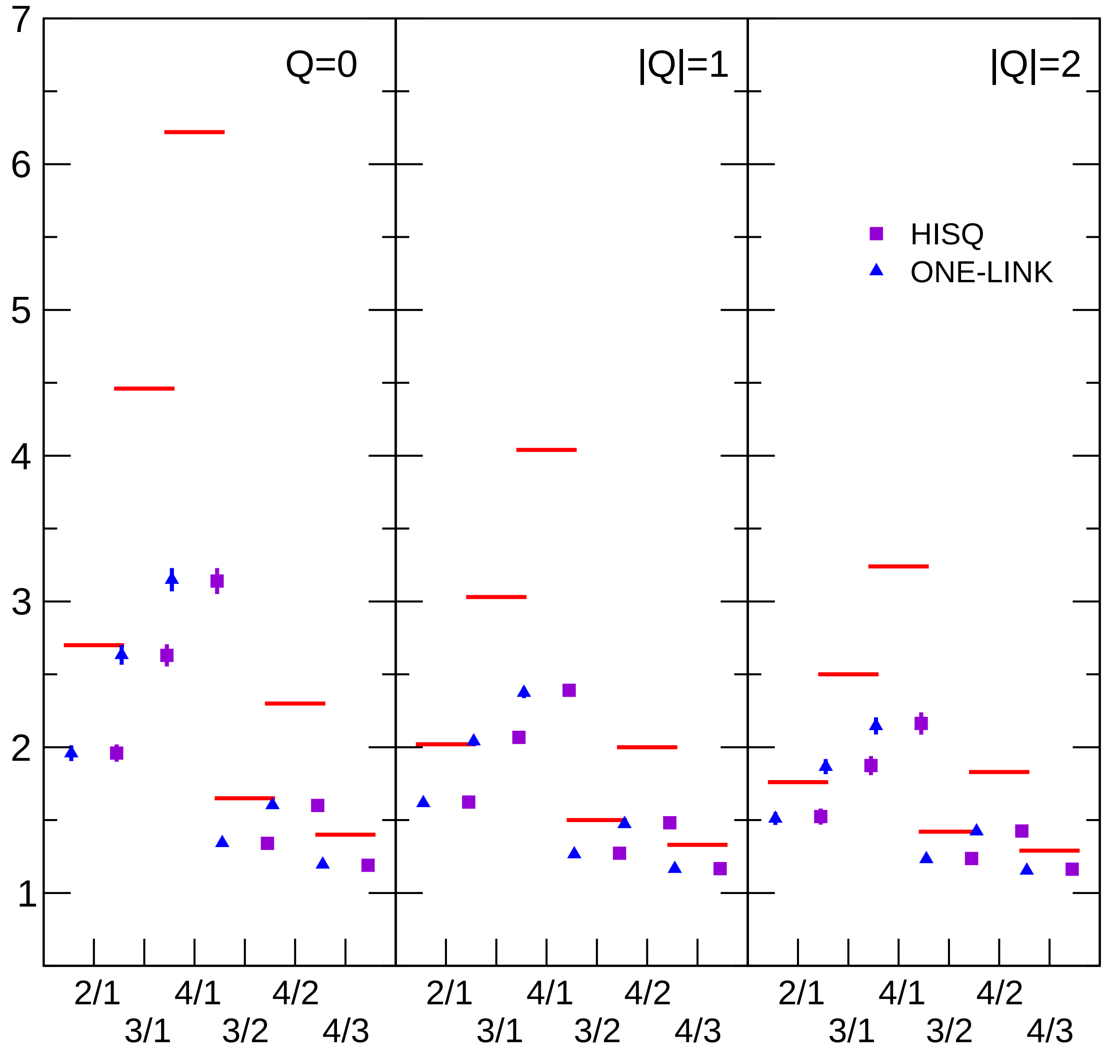
Eigenvalue Averages

Tree level improved glue, $a = 0.093$ fm 16^4 lattice, $L = 1.488$ fm



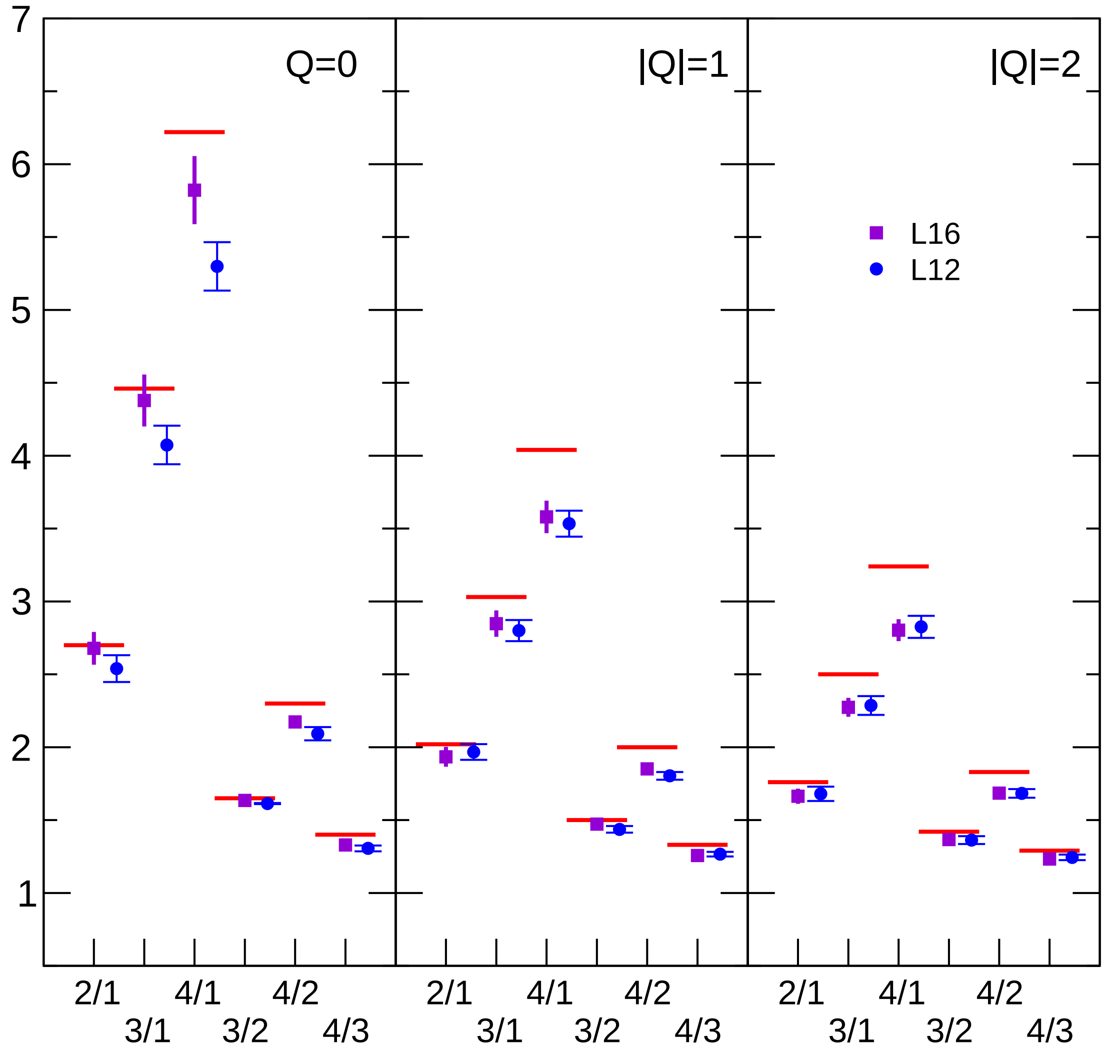
Eigenvalue Averages

Tree level improved glue, $a = 0.093$ fm 12^4 lattice, $L = 1.116$ fm



Eigenvalue Averages

Tree level improved glue, 12^4 $a = .125$ fm and 16^4 $a = .093$, approx same L.



Conclusions and Outlook

- Staggered quarks at values of the lattice spacing being used in current simulations are sensitive to topology.
- For improved staggered operators there is a sharp distinction between high and low chirality modes, and their respective number is in accordance with the Index Theorem.
- The predictions of RMT for the low-lying spectrum are not reproduced exactly for the lattices we have studied, but the results are quite close to the theoretical values, for each sector of Q.
- The spectrum degeneracy can be seen directly in some lattices, and can be also inferred from the comparison with RMT.
  This is a necessary condition to obtain a correct one-taste theory by taking the fourth root of the determinant.
- Study bigger lattice volume, to ascertain the influence of finite-size effects.